École doctorale de sciences mathématiques de Paris centre (386)

IMJ-PRG

Fractionally Calabi–Yau lattices

présenté par

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Thèse de doctorat

Discipline : Mathématiques

dirigée par Baptiste Rognerud et Bernhard Keller

Soutenue le 09 Décembre 2024 publiquement devant le jury composé de :

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People who deny the existence of dragons are often eaten by dragons. From within.

Ursula K. Le Guin, *The Wave in the Mind: Talks and Essays on the Writer, the Reader and the Imagination*

Remerciements

Je tiens tout d'abord à remercier mes directeurs de thèse. Merci à Baptiste Rognerud d'avoir non seulement dirigé cette thèse mais aussi de m'avoir montrer comment allier rigueur, enthousiasme et bienveillance d'une façon qui m'inspire chaque jour dans ma vie de jeune mathématicienne. Ce travail n'aurait pas vu le jour sans nos innombrables discussions, tes indications et ta relecture acérée. Merci à mon codirecteur Bernhard Keller. Comme prévu initialement, tu a été moins présent dans la direction de cette thèse mais ton regard à toujours été très juste au moment opportun. Sans vouloir divulgacher la suite, cette thèse est immensément redevable au travail Frederic Chapoton. C'est une joie immense de marcher dans tes pas. Merci d'avoir écouté mes exposés malgrè la gêne. Merci à Karin Baur de m'accorder un futur mathématique. J'ai hâte de travailler à tes côté. Merci enfin aux autre membres du jury: Hugh Thomas, Claire Amiot, David Hernandez et Emine Yıldırım. Vos travaux sont ce qui me donne envie de faire des mathématiques.

Si la vie d'un.e mathématicien.ne est faite de rencontre je me doit de remercier ici Daniel Juteau, dont l'ouverture mathématique m'a ouvert tant de portes. Je remercie par la même occation la FSMP dont le programme PGSM m'a offerte une indépendance et une stabilité sans lesquelles je n'aurai surement pas pu continuer les mathématiques [1]. Parmis les très nombreux professeur qui m'ont transmis leur passion pour les mathématiques je tiens aussi à remercier Olivier Dudas. Je me souviendrais tout particulièrement de ses cours en présentiel à la fin du deuxième confinement.

La suite du manuscript contient uniquement l'aspect mathématique des trois ans que j'ai passé à l'IMJ-PRG. Je me permets dans ces remerciements d'évoquer le reste. Merci aux membres de la commission pédagogique de l'UFR de mathématiques de l'Université Paris Cité. L'enseignement supérieur français repose sur des personnes comme vous. Je tiens tout particuliérement à évoquer et à remercier Séverine Leidwanger, présidente de la commission pédagogique aux multiples casquettes qui a su me poser les bonnes questions au bon moment. Merci aussi à mon tuteur de thèse Phillip Boalch. Merci aussi aux

[1] J'ai trouvé le dispositif de tutorat extrèmement pertinent et pense qu'il serait bénéfique à toustes les étudiant.e.s, pas seulement celleux qui reçoivent des bourses.

chercheureuses avec qui j'ai enseigné, Cyrille Lucas, Bertrand Gentou, Fanny Augeri, Pierre Fima, François Le Maitre. Vous m'avez beaucoup appris.

Merci aux doctorant.e.s de l'équipe GRG ou liée à feu l'ANR CHARMS: Jule, Esha, Riccardo, Mickael et maintenant Lior et Marius. J'ai beaucoup aimé partager mes maths avec vous. Merci aussi aux ami.e.s que je me suis faite ou refaite en conférences, Uriel, Luca et Eva. J'espère vous revoir bientôt. In english I will also mention Duncan Laurie, another conference buddy I was happy to meet at the university of Bath. Merci aux nombreureuses doctorant.e.s du SG. Ceux qui étaient la quand je suis arrivé.e et qui m'ont accueillie, Jeremy, Mingkun, Pierre, Dorian, Juan et Tomaso. Celleux qui sont arrivé.e.s en même temps que moi: Laura, Skander, Théo et Sergio. Et celleux qui sont arrivé.e.s après et qui pour beaucoup se sont impliqué.e.s à fond dans le BDD: Juan Ramon, Francesca, Vincent doc, Mathieu, Corentin. Et bien sûr, comment ne pas remercier Jerome, point cardinal de la vie sociale des doctorant.e.s, toujours la et toujours prêt à discuter. Merci à Fabien de m'avoir fait découvrir des groupes de Kpop, à Eli pour ses connaissances mathématiques et sa gentillesse sans fin, à Vincent post doc pour ses questions, à Merlin pour tous les merveilleux thés qu'il nous à fait déguster et à Mario dont la bonne humeur est contagieuse. Merci aux représentants du bureau 654, Alexandre avec ton emploi du temps de ministre parce qu'il est impossible de ne pas vouloir être ton ami, Leonardo avec ton coeur en or et William avec ton appétit vorace qui s'étend bien au delà de la nourriture. Merci à mon cobureau de toujours, tout à la fois critique musical, partenaire de ragot, roi des emails et ami, Arthur. Le hasard des clés a décidément bien fait les chose. À l'UPMC crew, Andres, Paul et Diego, merci de prendre soin de la franchise pizza five avec moi. Merci à mes amies de longues dates, Zoé, Louise et Lauréana, parce que je sais qu'on se *re*trouvera toujours. Clara, merci pour ta sensibilité, ton humilité à toute épreuve et la chemise que tu m'a cousue. Katy, my college sister, thank you for all the tea, the biscuits the plants and the talks. Célia, je suis toujours reconnaissante du temps que tu m'accordes alors que tu est occupée à gérer trois carrières à temps plein. Pépita, merci pour tous les bubble teas, les fous rire, et le monde que tu sauves.

To my family, I may be far away, and incapable of using the correct alphabet in this document, but you are always on my mind and I love you. A mes adelphes Maya, Daniel et David, vous voir grandir m'apporte une joie énorme. David, je m'adresse a toi en français parce que je sais que tu le liras très vite. Maman, Papa, évidemment, merci.

Gabriel merci de me comprendre quand je ne me comprend pas moi même.

Résumé long

Ce résumé est une traduction depuis l'anglais de certains passages de l'introduction ainsi que des passages des introductions de chacun des chapitres.

Cette thèse étudie la propriété Calabi–Yau fractionnaire, qui ici, aura trait à la théorie des représentations, appliquée à des ensembles ordonnés, en considérant des exemples liés à la Théorie de Lie de plusieurs façon différentes. Les ensembles ordonnés apparaissent naturellement dans beaucoup de domaines des mathématiques. Depuis l'ordre sur les entier relatifs jusqu'à l'optimisation des produits de matrices, les ordres nous permettent d'organiser des objets souvent de manière canonique, bien que rarement unique. La théorie des représentations est une branche des mathématiques qui traduit des structures algébriques abstraite en algèbre linéaire. Son exploration débute au dix-neuvième siècle avec l'étude des groupes symétriques agissant sur des ensembles. A travers l'algèbre d'incidence d'un ensemble ordonné, on peut appliquer les idées fécondes de la théorie des représentations aux ensembles ordonnés. Simultanément, l'algèbre d'incidence d'un ensemble ordonné fini a des propriétés qui rendent sa théorie des représentations plus simple à décrire combinatoirement. De nombreux objets centraux de la théorie des représentations sont apparus d'abord en théorie de Lie. Il s'agit d'une vaste branche des mathématiques à l'intersection de la géométrie, de la théorie des représentations et de la résolution d'équations différentielles partielles dont une motivation centrale est de décrire des phénomènes physiques. Dans les années mille neuf-cent quarante, des objets au centre de la théorie de Lie, les groupes de Lie compacts et le algèbres de Lie semi simples, ont été classifiés à l'aide de systèmes de racines que l'ont peut encoder par des diagrammes dit de Dynkin. La propriètè Calabi–Yau fractionnaire vient d'un autre domaine de la physique et exprime l'absence de courbure de variétés qui apparaissent en relativité générale.

0.1 Prolégomène

Dans le Chapitre 2 on présente le contexte nécessaire à la compréhension du manuscript, c'est à dire la théorie des représentations des ensembles ordonnés.

La Section 2.1 permet d'évoquer les fondements algébriques et combinatoires de celle-ci. On commence par fixer des notations et des conventions de la théorie des représentations des carquois, dans laquelle notre travail sur les algèbres d'incidence d'ensembles ordonnés s'inscrit naturellement. On évoque ensuite quelques notions de la théorie d'Auslander–Reiten. On termine cette section par une discussion des différentes notions de dimension d'un ensemble ordonné. Les mathématiques présentées dans cette section dâtent au plus tard des années 1970. Depuis, de nombreux progrès ont émergé au sein de chacun des domaines évoqués. Les Sections 2.2 et 2.3 relatent certains de ces progrès.

Dans la Section 2.2 on rappel la construction de la catégorie dérivée d'une catégorie abélienne, sa structure triangulée, puis on discute de foncteurs importants. Les résultats de cette section remontent au plus tard aux années 1990. Un lemme classique sur les morphismes de triangles sera utilisé de façon cruciale dans la preuve du Théorème principal du Chapitre 3.

La Section 2.3 est une courte présentation de la théorie d'Auslander supérieure qui à été développée dans les années 2000. Notre but est de d'introduire l'algèbre d'Auslander supérieure de type A qui joue un rôle centrale dans les résultats du Chapitre 5.

L'auteure est temtée de dire que le passage à la catégorie dérivée et la théorie d'Auslander supérieure sont deux façons d'étendre la théorie des représentations classique. On peut dire que la première est devenue nécessaire lorsqu'on a voulu remplacer les modules par leurs résolutions projectives alors que la seconde répond à des questions naturelles comme *"que ce passe-t-il lorsque l'on remplace un 1 (caché dans certaines formulations) par un n"*. Cependant, la théorie d'Auslander supérieure à été développée après les aspects dérivés de la théorie des représentations et donc est fermement ancrée dans ce contexte.

Certains aspects de la théorie des représentations contemporaine n'apparaissent pas dans ce manuscript, comme l'importance de la théorie des amas, le fait que les catégories triangulées sont maintenant souvent remplacées ou renforcées par des structures plus robustes que sont les dg–catégories, et l'importance croissante que prend le language des infini-catégories.

Enfin, dans la Section 2.4, on décrit la conjecture de Chapoton qui motive une grande partie du travail présenté dans cette thèse.

0.2 Résultats

Cette thèse contient deux types de résultats. D'une part, nous présentons des résultats qui ont traits soit aux treillis finis en toute généralités, soit aux treillis join-semiditributifs, en espérant que ces résultats puissent servir d'outils à d'autre chercheureuse. Dans un deuxième temps, nous appliquons ces résultats à l'étude d'une famille de treillis particulière dans le but de corroborer une conjecture de Chapoton. Au coeur de l'ensemble de ces résultats, il y a l'étude des antichaines dans les treillis et des représentations qu'elles décrivent

0.2.1 Dimension globale de treillis join-semidistributifs

Les antichaines apparaissent naturellement dans la théorie des représentations des ensembles ordonnés car elles permettent de décrire les sous modules des modules projectifs indécomposables et de là, les modules à tête simple [37]. Lorsque l'ensemble ordonné est un treillis, ces modules à tête simple admettent une resolution projective canonique [37] mais pas toujours minimale, décrite par les sous ensembles de l'antichaine correspondante. Dans le Chapitre 3 nous identifions quatre types d'antichaines : les antichaines *intersectives*, les antichains *inclusives*, les antichaines *fortes* et les antichaines *booléennes*. Nous établissons des liens entre ces propriétés (Lemme 3.1.1). Nous montrons qu'une antichaine est booléenne si et seulement si elle engendre un sous treillis qui est booléen, ce qui justifie ce choix de terminologie et constitue l'intérêt de ces antichaines. La Proposition 3.2.4 établie que la résolution canonique d'un module d'antichaine défini par une antichaine forte est une réolution minimale ce qui constitue à son tour, l'intérêt des antichaines fortes. Cela nous permet de démontrer le Corollaire 3.2.2 qui généralise un résultat de [37] sur les treillis distributifs aux treillis joints semi distributifs: leur dimension globale ne dépend pas du corps de base choisi.

0.2.2 Détection de catégories Calabi–Yau fractionnaires

La notion de catégorie Calabi–Yau fractionnaire fut introduite par Kontsevich à la fin des années 1990 [42, Definition 28]. Une catégorie triangulée $\mathcal{T}$ avec un foncteur de Serre $\mathbb{S}$ est dite Calabi–Yau fractionnaire s'il existe des entiers l et d telle que $\mathbb{S}^l$ est isomorphe comme foncteur au foncteur suspension de $\mathcal{T}$ appliqué d fois. Dans ce cas on dit que $\mathcal{T}$ est $\frac{d}{l}$-Calabi–Yau fractionnaire. Quand $\mathcal{T} = D^b(A)$, la catégorie dérivée bornée d'une algèbre de dimension fini A sur un corps $\Bbbk$, le foncteur de Nakayama dérivé, $\mathbb{S} = DA \otimes^{\mathbb{L}}_{A} ?$, est un foncteur de Serre. Dans ce cas, la propriété Calabi–Yau fractionnaire peut être étendue. Si ϕ est un automorphisme de A on dit que $D^b(A)$ est Calabi–Yau fractionnaire tordue quand $\mathbb{S}^l \simeq [d] \circ \phi^*$, où ϕ^* tord l'action de A sur un module par ϕ. Pour montrer qu'une algèbre est Calabi–Yau fractionnaire tordue, il suffit, de montrer que l'on a un isomorphisme

$$\mathbb{S}^l\, A \simeq A[d]$$

dans $D^b(A)$ [31, Proposition 4.3]. La question de savoir si toutes les algèbres de dimension globale finie Calabi–Yau fractionnaires tordues sont Calabi–Yau fractionnaires est ouverte [31, Remark 1.6]. Pour les ensembles ordonnés finis avec un unique maximum ou un unique minimum, le Théorème [51, Theorem 3.1] offre une réponse positive à cette question. Ce Théorème affirme en particulier, que la propriété Calabi–Yau fractionnaire peut se vérifier sur les modules projectifs indécomposables. Cependant, étant donnée l'algèbre d'incidence de l'ensemble ordonnés, il est en général très dur de vérifier l'isomorphisme $\mathbb{S}^l(A) \simeq A[d]$ [51][60]. Le résultat technique principal de cette thèse est le Théorème 3.5.1 qui se base sur, et relaxe le Théorème [51, Theorem 3.1] dans le contexte des treillis finis.

Theorem (B). *Soit L un treillis fini, soient d et l des entiers, et soit $(M_\alpha)_{\alpha \in L}$ une famille de modules d'antichaines fortes. Si pour tout $\alpha \in L$ on a l'isomorphisme $\mathbb{S}^l(M_\alpha) \simeq M_\alpha[d]$ alors L est $\frac{d}{l}$- Calabi–Yau fractionnaire.*

La preuve de ce théorème consiste à construire un isomorphismes $\mathbb{S}^d(P) \cong P[l]$ pour chaque module projectif indécomposable, à partir des isomorphismes $\mathbb{S}^d(\mathcal{P}_C) \cong \mathcal{P}_C[l]$ qu'on a sur la famille des modules d'antichaines fortes, pour pouvoir appliquer le Théorème [51, Theorem 3.1]. Pour cela, on effectue un raisonnement par récurrence forte sur les éléments α du treillis, qu'on appel récurence externe. L'idée phare de la preuve est de construire des isomorphismes entre des objets gradués, les troncations des résolutions d'antichaines du modules M_α et leurs images respectives par le foncteur $\mathbb{S}^d[-d]$, à partir d'isomorphismes en chaque degrés. Ceci prend la forme d'un deuxième raisonnement par récurence, qu'on appelle la récurence interne. La nature forte des antichaines est cruciale. Les Lemmes 3.4.1 et 3.4.4 décrivent les morphismes entre les résolutions d'antichaines fortes et leurs troncations. Ces lemmes nous permettrons de *rectifier* certains diagrammes de morphismes de complexes de chaines pour qu'ils commutent et qu'on puisse appliquer l'axiome **TR3** et le lemme des "2 parmi 3" et obtenir l'isomorphisme voulu.

0.2.3 Un exemple qui corrobore une conjecture de Chapoton

Les ensembles ordonnés Calabi-Yau fractionnaire font l'objet d'une conjecture fascinante due à Frédéric Chapoton [14] reliants des suites d'entiers d'origine combinatoire á des catégories de Fukaya en passant par les ensembles ordonnés. Certaines suites d'entiers $(s_n)_{n\in\mathbb{N}}$ connues peuvent s'écrire sous la forme d'un produit de fractions $s_n = \Pi_{i=1}^n \frac{D_n - d_i^n}{d_i^n}$ où la somme d'un dénominateur avec son numérateur respectif est égale à une constante D_n. C'est le cas des nombres de Catalan, du nombre des matrices à signes alternés, de la famille de West et des intervalles de Tamari. La conjecture de Chapoton propose d'expliquer cette coincidence par l'existence d'ordres partiels sur des ensembles P_n de cardinal s_n dont les catégories dérivées respectives seraient $\frac{C_n}{D_n}$-Calabi–Yau où

$$C = \sum_i D_n - 2d_i.$$

De plus, la catégorie dérivée bornée en question serait équivalente a une catégorie de Fukaya Seidel d'une surface construite à partir de la donnés des entiers D_n et des coefficients $d_1^n, \ldots, d_i^n$. La conjecture prédit aussi une formule pour le polynôme de Coxeter et le nombre de Milnor de la surface ainsi trouvée, qui peuvent faire l'objet de vérifications par ordinateurs. Certaines conséquences de cette conjecture ont été démontrées [51].

Le but initiale de cette thèse était de démontrer une autre de ces conjectures résultantes dont l'étude avait en partie été faite dans [60]. On remarque que le coefficient binomial $\binom{m+n}{m}$ peut s'écrire comme suit

$$\frac{m+n}{1}\frac{m+n-1}{2}\cdot\cdots\cdot\frac{m+1}{n} \tag{1}$$

où $D = m+n+1$. Ceci est probablement l'un des examples les plus naturels de formule produit. Le treillis des idéaux d'ordre du produits de deux ordres totaux de taille m et n, qu'on note ici $J_{m,n}$, est de cardinal $\binom{n+m}{m}$. Avec nos résultats sur les modules d'antichaines fortes nous somme en mesure de confirmer la prédiction de Chapoton sur la dimension Calabi–Yau fractionnaire de ces ensembles ordonnés.

Theorem (C). *La catégorie dérivée bornée de $J_{m,n}$ est $\frac{mn}{m+n+1}$-Calabi–Yau.*

Un corollaire directe donne une réponse positive à la conjecture de Chapoton–Yıldırım sur les ensembles ordonnés cominuscules de type A et B.

Corollary (D). *La catégorie dérivée bornée des ensembles ordonnés cominuscules de type A, B or D est Calabi–Yau fractionnaire.*

La preuve du théorème précédent consiste à appliquer le théorème B à une famille de modules d'antichaines bien choisie. Dans le Chapitre 4 on considère donc une famille trouvée et étudiée dans [60] que l'on note $(\mathcal{P}_\alpha)_{\alpha\in J_{m,n}}$. Pour que cette thèse contienne tous les éléments nécessaires à la compréhension des résultats principaux, on reproduit certaines preuve de [60] en les adaptants à nos notations. Nous en déduisons que ces modules d'antichaines satisfont les conditions du Théorème B et prouvons ainsi notre deuxième résultat notable.

Le dernier résultat notable de cette thèse fourni une raison plus structurelle au fait que les catégories dérivées bornées de ces ensembles ordonnés soient Calabi–Yau fractionnaires.

Theorem (E). *L'algèbre d'incidence de $J_{m,n}$ est dérivée equivalente à l'algèbre d'Auslander supérieure A_{m+1}^{n-1}.*

La combinaison des théorèmes C et E donne une preuve nouvelle du fait que ces algèbres sont Calabi–Yau fractionnaires [22][21][28]. De plus, il faut noter que cette equivalence dérivée semble corroborer l'aspect géométrique de la conjecture de Chapoton dans le cadre de la famille d'examples étudiée ici. En effet, il existe une catégorie de Fukaya enroulée associée aux Algèbres d'Auslander supérieures de type A [21]. Une prépublication plus récente lui associe même une catégorie de Fukaya-Seidel dont le nombre de Milnor est celui prédit Chapoton.

La preuve du Théorème E repose sur un résultat spécifique aux antichaines booléennes (Theorem 3.3.4): les espaces d'homomorphismes entre un module d'antichaine boolénne et un interval décalé sont au plus de dimension un et sont concentrés en un seul degrés. Dans le Chapitre 5 on se sert de ce résultat pour décrire les morphismes de la sous catégorie pleine de $D^b(J_{m,n})$ donc les objets sont les $\mathcal{P}_\alpha$ et leur shifts. On la note $\mathcal{Y}_{m,n}$. On décompose les morphismes en morphismes irréductibles (Lemme 5.1.13, Lemme 5.1.14) et on identifie les relations entre ceux-ci (Lemme 5.2.3, Proposition 5.2.12). Ainsi, on peut trouver un objet basculant (Proposition 5.3.1, Lemme 5.3.3) dont l'algèbre des endomorphismes est isomorphe à A_{m+1}^{n-1} mais aussi à son dual quadratique $(A_{n+1}^{m-1})^\dagger$. Ceci conclu la preuve du Théorème E tout en établissant un isomorphisme explicite entre A_{m+1}^{n-1} et $(A_{n+1}^{m-1})^\dagger$. On fini cette thèse en donnant une nouvelle charactérisation des morphismes de la catégorie $\mathcal{Y}_{m,n}$ en terme de *suites entrelacées* (Proposition 5.4.9), d'une façon qui fait echo à un résultat similaire pour les algèbre d'Auslander supérieures [31].

Résumé court

Dans cette thèse nous étudions la propriété Calabi-Yau fractionnaire sur des treillis. Nous commençons par introduire des familles combinatoires de représentations de treillis qui ont de bonnes propriétés homologiques. Nous en déduisons un premier résultat qui généralise aux treillis join-semidistributifs un théorème de Iyama et Marczinzik sur la dimension global d'un treillis distributif. Ensuite nous montrons que la propriété Calabi-Yau fractionnaire peut être vérifiée sur des familles de représentations au bon comportement pour les treillis finis. De plus nous donnons un critère combinatoire pour calculer certains espaces de morphismes. Nous utilisons ces deux résultats pour montrer que la catégorie dérivée du treillis des idéaux du produit de deux chaines est Calabi-Yau fractionnaire. On établi de plus une équivalence dérivée entre ces treillis et les algèbres d'Auslander supérieures de type A et une autre avec leur dual quadratiques. Ces deux résultats corroborent une conjecture de Chapoton qui relie des ensembles ordonnés à des catégories de Fukaya-Seidel tout en donnant des isomorphismes d'algèbre entre des objets déjà connus. On décris les catégories en question avec des suites entrelacées.

Mots-clés

Théorie des représentations, algèbre de dimension finie, ensemble ordonné.

Fractionally Calabi-Yau Lattices

Abstract

In this thesis we study the fractionally Calabi-Yau property in lattices. We start by introducing combinatorial families of lattice representations with good homological properties. As a first result we extend a Theorem of Iyama and Marczinzik on the global dimension of distributive lattices to join semi distributive lattices. We then prove that the Calabi-Yau property can be checked on well behaved families on modules for finite lattices. We also give combinatorial criteria to compute certain hom spaces. We use these two results to show that the bounded derived category of the lattice of order ideals of the product of two ordered chains is fractionally Calabi-Yau. We also show that these lattices are derived equivalent to higher Auslander algebras of type A as well as their quadratic duals. These two results combined corroborate a conjecture by Chapoton linking posets to Fukaya Seidel Categories and at the same time gives interesting algebra isomorphisms between well known objects. We give a description of the categories at hand using interlacing sequences.

Keywords

Representation theory, finite dimensional algebras, posets.

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Chapter 1

Introduction

This thesis studies the Calabi–Yau property, which here will be representation theoretical in nature, on partially ordered sets, considering examples linked in several ways to Lie theory. Partially ordered sets, or posets, are ubiquitous in mathematics. From the way that we count natural numbers to the optimisation of the products of matrices, orders help us organise objects and processes, in canonical albeit non unique ways. Representation theory is a branch of mathematics that translates algebraic structures to linear algebra. It arose in the nineteenth century through the study of symmetries acting on sets. Using the incidence algebra of a poset, we can apply the fruitful ideas of representation theory to the study of posets. At the same time, incidence algebras of finite posets possess properties that make their representation theory easier to describe combinatorially. Many central objects in representation theory arise from Lie theory. It is a broad area of mathematics which lies at the intersection of geometry, representation theory and the resolution of partial differential equation with the strong motivation of describing physical phenomena. In the nineteen-forties, it became apparent that central objects of Lie theory, *i.e.* compact Lie groups and semi simple Lie algebras, were classified by root systems summed up using Dynkin diagrams. The fractionally Calabi–Yau property comes from another field of physics and expresses flatness of manifolds which arise in general relativity. We start by giving an overview of the different notions mentioned above and list our results before setting some notation and giving a detailed outline of the subsequent chapters.

The notion of fractionally Calabi–Yau categories was introduced by Kontsevich in the late nineteen-nineties [42, Definition 28]. A triangulated category $\mathcal{T}$ with a Serre functor $\mathbb{S}$ is said to be *fractionally Calabi–Yau* if there exists l and d such that $\mathbb{S}^l$ is isomorphic as a functor to the suspension functor applied d times. We say that $\mathcal{T}$ is $\frac{d}{l}$-Calabi–Yau. When $\mathcal{T} = D^b(A)$, the bounded derived category of an algebra A of finite global dimension over a field $\Bbbk$, we can take $\mathbb{S} = DA\otimes^{\mathbb{L}}_{A}?$ the derived Nakayama functor. In that case, the Calabi–Yau property can be further relaxed. If ϕ is an automorphism of A, then $D^b(A)$, is said to be twisted fractionally Calabi–Yau if $\mathbb{S}^l \simeq [d] \circ \phi^*$, where ϕ^* twists the action of A on a module by ϕ. We recover the previous definition when $\phi = id_A$. The following theorem makes it easier to detect twisted fractionally Calabi–Yau algebras.

Theorem A ([31, Proposition 4.3])**.** *Let Λ be a finite dimensional $\Bbbk$-algebra of finite global dimension. The following conditions are equivalent.*

(i) Λ *is twisted* $\frac{d}{l}$*-Calabi–Yau.*

(ii) $\mathbb{S}^l\,\Lambda \simeq \Lambda[d]$.

This leads to a question which is still far from being answered in general.

Question 1 ([31, Remark 1.6])**.** *Is every twisted fractionally Calabi–Yau algebra fractionally Calabi–Yau?*

Because the trivial extension algebra of a (twisted) fractionally Calabi–Yau finite dimensional algebra of finite global dimension is (twisted) periodic [11], Question 1 is linked to the following conjecture of Erdmann and Skowroński [23].

Question 2 ([11, Question 1.4])**.** *Is every finite-dimensional twisted periodic algebra periodic?*

In the case of finite posets with a unique maximal element or a unique minimal element, the answer to Question 1 is positive as per [51, Theorem 3.1]. However, for a given incidence algebra the existence of an isomorphism

$$\mathbb{S}^l(A) \simeq A[d]$$

is still in general very hard to check [51][60]. In this thesis we provide a relaxation of [51, Theorem 3.1], in the context of finite lattices, to help overcome that difficulty.

Theorem B. *Let L be a finite lattice, d and l integers and $(C_\alpha)_{\alpha\in L}$ be a family of indecomposable modules with simple head S_α and having a* boolean resolution*. If for all $\alpha \in L$ it holds that $\mathbb{S}^l(C_\alpha) \simeq C_\alpha[d]$, then L is $\frac{d}{l}$- fractionally Calabi–Yau.*

This theorem does not provide with a recipe to find appropriate families, but it suggests certain criteria which restrict the search for good candidates.

Fractionally Calabi–Yau posets are fascinating objects in part due to a hypothetical relation to *product formulas* due to Chapoton [14]. In combinatorics, many families of sets $(S_n)_{n\in\mathbb{N}}$ can be counted by a product of fractions $|S_n| = \Pi_{i=1}^{n} \frac{D-d_i}{d_i}$ where the sum of the numerator and denominator is constant and equal to D. Such families include the Catalan numbers, the number of alternating sign matrices, the West family and the Tamari intervals family. Chapoton's highly conjectural explanation is that there should exist a partial order on S_n whose derived category is $\frac{C}{D}$-Calabi–Yau, where

$$C = \sum_i D - 2d_i.$$

Moreover, the bounded derived category in question should be equivalent to a type of Fukaya-Seidel category associated to a singularity constructed from the data of D and the d_i coefficients. The conjecture also provides predictions regarding the Coxeter polynomial and the Milnor number of the singularity some of which can be tested with a computer on examples.

Some consequences of these conjectures have been proven since [51]. The starting point of this thesis was to prove another one of these resulting conjectures which was already studied in part in [60]. Observe that the binomial $\binom{m+n}{m}$ can be written as

$$\frac{m+n}{1}\frac{m+n-1}{2}\cdot\ldots\cdot\frac{m+1}{n} \tag{1.1}$$

where $D = m + n + 1$. This is probably one of the most natural examples of product formulas discussed above. The poset of order ideals of a product of total orders of length m and n has cardinality $\binom{n+m}{m}$ and we write it $J_{m,n}$. Using our results on *boolean antichain modules* we are able to confirm Chapoton's prediction about the Calabi–Yau dimension of these posets.

Theorem C. *The bounded derived category of $J_{m,n}$ is $\frac{mn}{m+n+1}$-Calabi–Yau.*

As a corollary this gives a positive answer to the Chapoton-Yıldırım conjecture on cominuscule posets of type A and B [60].

Corollary D. *The bounded derived category of the poset of order ideals of a cominuscule poset of type A or B is fractionally Calabi–Yau. For types A and B, the denominator is $h+1$ where h is a constant associated with the root system.*

The key observation one needs for applying Theorem C to cominuscule posets is their classification into types C_I, C_{II} or C_{III} depicted below [56]. Interestingly, there is no one to one correspondence between this classification and the ADE classification of the root posets one started with. However cominuscule posets of type A and B follow the pattern

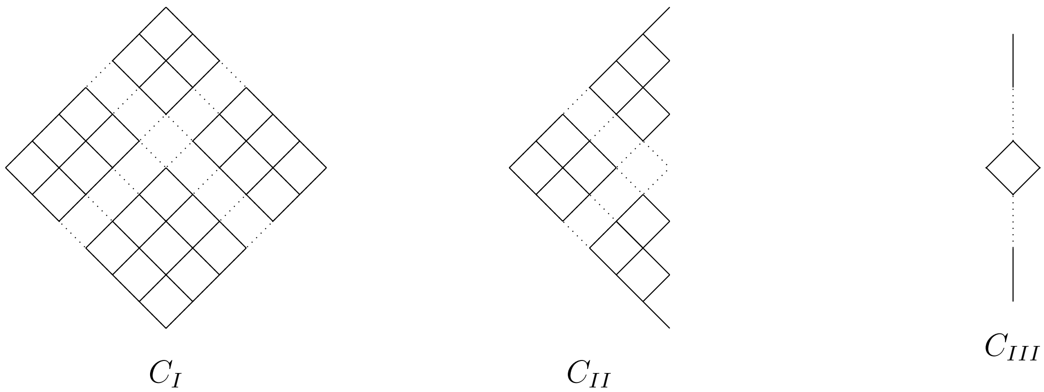


Figure 1.1: The three types of cominuscule posets

C_I which is in particular, a grid. Corollary D follows from that. Type D follows patterns C_{III} or C_{II}. Type C follows pattern C_{II}. Pattern C_{III} is studied using Ladkani's flip flop technique [43], [60]. Pattern C_{II} seems to be different and is not a consequence of our work. The conjecture is still open in this case

Our proof of Theorem C gives a good understanding of the Serre functor for this category. However, one would like to have a more structural reason behind the fractionally Calabi–Yau property for posets. For us, a good reason why $J_{m,n}$ should be fractionally Calabi–Yau is our second main result which is the following derived equivalence.

Theorem E. *The algebra of the poset $J_{m,n}$ is derived equivalent to the higher Auslander algebra A^{n-1}_{m+1}.*

Higher Auslander algebras were introduced by Iyama in [33] as part of a series of seminal articles on higher representation theory. Higher Auslander algebras of type A were soon after described in [35] and are known to be fractionally Calabi–Yau. As a corollary of Theorem C and Theorem E we have a new proof of an already known theorem.

Theorem F. *Higher algebras of type A are fractionally Calabi–Yau.*

Previous proofs of this result have different flavours. The first stemmed from symplectic geometry [21], the second, from the theory of infinity categories [22] and the most recent from an intricate study of the properties of a certain preprojective algebra and its Nakayama automorphism, linking it to the the Serre functor [28]. The proof presented here is more combinatorial. Of course knowing Theorems E and F also gives a proof of Theorem C. It is also satisfying to note that this derived equivalence ties back into Chapoton's conjectures: a partially wrapped Fukaya category can be associated to higher Auslander algebras of type A [21]. A more recent preprint [16] also links the higher Auslander algebras of type A to Fukaya–Seidel categories with the Milnor number predicted by Chapoton.

1.1 Notation

Generalities Let $\Bbbk$ be a field and X a finite partially ordered set or poset. Define its incidence algebra $A = A_{\Bbbk}(X)$ over $\Bbbk$ to be the $\Bbbk$-vector space with basis pairs (x, y) such that $x \leq y$ with multiplication defined by

$$(x,y)(z,t) = \begin{cases} (x,t) & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in X$, we write $e_x = (x, x)$ the primitive idempotent. Then the unit of the algebra A is $1_A = \sum_{x \in X} e_x$. Throughout this work we consider finite dimensional left modules over A. For every element $x \in X$ the associated simple module is $S_x \cong \Bbbk$ with action $(y,t) \cdot 1_{\Bbbk} = 0$ unless $y = t = x$ in which case $e_x \cdot 1_{\Bbbk} = 1_{\Bbbk}$. Its projective cover $P_x = A \cdot e_x$ has basis $\{(y,x) | y \leq x\}$. Its injective hull is the injective indecomposable $I_x = (e_x \cdot A)^*$ and has basis $\{(x,y)^* | x \leq y\}$. Morphisms between the projective indecomposable modules are characterised by

$$\mathrm{Hom}_A(P_x, P_y) = \mathrm{Hom}_A(Ae_x, Ae_y) \cong \begin{cases} e_x A e_y \cong \Bbbk & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the canonical inclusion as $\iota_x^y : P_x \hookrightarrow P_y$ whenever $x \leq y$ which is the right multiplication by (x, y). More generally for any left A-module M, for all $x \in X$ we have $\mathrm{Hom}_A(P_x, M) \cong e_x M$. This isomorphism makes the following diagram commute

$$\begin{tikzcd} f \arrow[d, mapsto] & \in & \mathrm{Hom}_A(P_x, M) \arrow[d] & \mathrm{Hom}_A(P_y, M) \arrow[l, "\circ \iota_x^y"'] \arrow[d] & \ni & g \arrow[d, mapsto] \\ f(e_x) & \in & e_x M & e_y M \arrow[l, "(x,y)\cdot"] & \ni & g(e_y) \end{tikzcd} \tag{1.2}$$

The total hom complex $\mathrm{Hom}_A^\bullet(C, M)$ where C is a chain complex $C = ((C_n)_n, (\partial_n))$ of A-modules and M is an A-module, is the complex

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(C_n, M) \xrightarrow{\partial_{n+1}^*} \mathrm{Hom}_{\mathcal{A}}(C_{n+1}, M) \to \ldots.$$

Note that we omit a conventional sign for the boundary map as it plays no role in our computations. This is a cochain complex as the functor $\mathrm{Hom}_A(-, M)$ is contravariant. Assuming that $C_n = \bigoplus_{x \in S_n} P_x$ with S_n a finite multi-subset of X and taking its cohomology gives shifted morphisms in the homotopy category $\mathrm{Ho}(A\text{-mod})$ [61, Lemma 3.7.10], which in turn are isomorphic to the shifted morphism in the derived category because the source is a perfect complex:

$$\mathrm{H}^i(\mathrm{Hom}^\bullet_A(C, M)) \cong \mathrm{Hom}_{\mathrm{Ho}(A\text{-mod})}(C, M[i]) \overset{u}{\cong} \mathrm{Hom}_{D^b}(C, M[i]). \tag{1.3}$$

Most computations will be carried out explicitly in the homotopy category. When needed the switch from one to the other will be discussed. Finally, using equation (1.2) we have an isomorphism of cochain complexes

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{Hom}_A(\bigoplus\limits_{x \in S_n} P_x, M) \arrow[r, "\partial^*_{n+1}"] \arrow[d] & \mathrm{Hom}_A(\bigoplus\limits_{x \in S_{n+1}} P_x, M) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \bigoplus\limits_{x \in S_n} e_x M \arrow[r] & \bigoplus\limits_{x \in S_{n+1}} e_x M \arrow[r] & \dots \end{tikzcd} \tag{1.4}$$

The boundary maps of the bottom complex are linear combinations of left multiplication by elements (x, y) of the algebra with coefficients inherited from the top complex.

Antichain Modules Let $(L, \wedge, \vee)$ be a finite lattice. We write $\hat{1}$ its greatest element and $\hat{0}$ its least one. Let C be an *antichain* in L *i.e.* a subset C of L that consists of pairwise incomparable elements of L. We say an antichain C is below an element α of L if for all $c \in C$, we have $c \leq \alpha$, and when needed we record this information by the notation C_α. Following [37, Proposition 2.1] we associate to an antichain $C = \{c_1, \dots, c_r\}$ the submodule

$$N_C := \sum_{i=1}^{r} A \cdot (c_i, \hat{1})$$

of the projective indecomposable $P_{\hat{1}}$ generated by the antichain. It follows directly from the same proposition that there is a one to one correspondence between antichains and submodules of $P_{\hat{1}}$. The *antichain module* associated to C is defined by

$$M_C := P_{\hat{1}} / N_C.$$

We will talk of antichain modules below $\alpha \in L$ by restricting to the sublattice $[\hat{0}, \alpha]$ of L. Then α is the greatest element of this lattice and there is a bijection between submodules of P_α and antichains below α. The corresponding modules will be denoted N_C^α and M_C^α. As our main example consider $a \le b$ in L. The maxima of the set of elements of L which are strictly less than b but not above a form an antichain C and the antichain module below b associated to C has support the interval $[a, b]$. The corresponding antichain module is usually called an *interval module*. In the rest of the paper we identify intervals with their interval modules.

Lemma 1.1.1. *Intervals are antichain modules.*

With the convention of the previous paragraph, morphisms between interval modules follow a simple rule

$$\mathrm{Hom}_A([a,b],[c,d]) = \begin{cases} \Bbbk & \text{if } a \le c \le b \le d, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

By [37, Theorem 2.2], for every antichain C of cardinal r of a lattice L the associated antichain module M_C has a projective resolution $\mathcal{P}_C$ of the form

$$0 \to P_r \to \cdots \to P_0 \to M_C \text{ where } P_0 = P_{\hat{1}} \text{ and } P_l = \bigoplus_{\substack{S \subseteq C \\ |S| = l}} P_{\wedge S} \text{ for } 1 \le l \le r.$$

Similarly, define a resolution $\mathcal{P}_C^\alpha$ for the antichain module M_C^α below α by replacing $P_{\hat{1}}$ by P_α. The boundary maps are defined by fixing an arbitrary total ordering of elements in C and, in each degree, setting the following maps between the indecomposable summands of the source and target in each degree:

$$\begin{array}{ccc} P_{\wedge S} & \to & P_{\wedge T} \\ (x, \wedge S) & \mapsto & \begin{cases} (-1)^{|i|_S}(x, \wedge T) & \text{if } T \sqcup \{i\} = S, \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{1.6}$$

for each $S = \{i_1, \ldots, i_k\}$ and $\big(\wedge S, \wedge T\big) \in P_{\wedge T}$ where $|i|_S = |\{j \in S | j \le i\}|$.

1.2 Detailed outline

In Chapter 2 we present background on the representation theory of posets. We revisit some of the objects discussed in the Notation section.

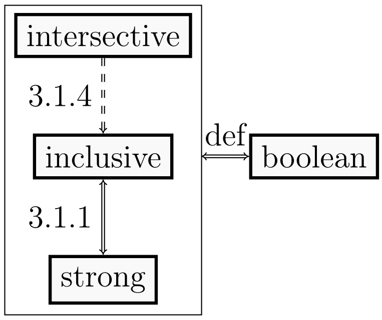


Figure 1.2: Properties of Antichains

In Chapter 3 we introduce four properties on antichains in lattices: *intersectivity*, *inclusivity*, *strength* and *booleanity*. They are related as in figure 1.2[1]. In Lemma 3.1.3 we show that an antichain C is boolean if and only if it spans a boolean sublattice in L. This justifies the terminology. Boolean antichains have a crucial property (Theorem 3.3.4): hom spaces in the homotopy category from a *boolean* antichain module to a shifted *interval* are at most one dimensional and are concentrated in one degree. This does not hold for antichains that are only strong.

However, certain hom spaces can still be controlled well enough. More specifically Lemmas 3.4.1 and 3.4.4 describe the maps between a resolution of a *strong* antichain module and its truncations.

Using these lemmas we prove Theorem 3.5.1 which is the main technical result of this thesis. It is a categorification theorem that builds upon and broadens [51, Theorem 3.1] which states that it suffices to check the Calabi–Yau property on projective indecomposable modules in a finite poset with a least or greatest element. Our result implies that, for finite lattices, it suffices to check Calabi–Yau property on any family of non zero strong antichain modules as long as it is sufficiently large. The proof consists in constructing isomorphisms $\mathbb{S}^d(P) \cong P[l]$ for projective indecomposable modules using the isomorphisms $\mathbb{S}^d(\mathcal{P}_C) \cong \mathcal{P}_C[l]$ on the family of antichain modules. We proceed by strong induction on the elements of the lattice. The main idea of the proof is to construct an isomorphism between graded objects out of ismorphisms in each degree. This requires a so called *inner induction*. See Figure 1.3 for an illustration. The fact that these objects come from strong antichains will be crucial as we use Lemmas 3.4.1 and 3.4.4 to ensure that required squares commute by *rectifying* any discrepancy. As a result we can apply axiom **TR3** and the 2 out of 3 Lemma 3.5.3 and gain the isomorphism we want.

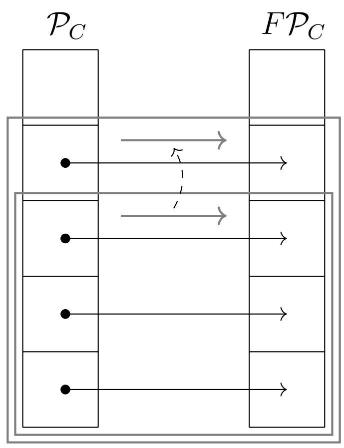


Figure 1.3: Isomorphisms of graded objects out of homogeneous one.

Theorem 3.5.1 is applied in Chapter 4 to the incidence algebra of the lattice of order

[1]The dashed arrow refers to a late addition to the manuscript

ideals of the product of two ordered sets. To discuss antichains it is convenient to see the elements of this lattice as paths in grids. We consider a family of antichains introduced in [60]. Its associated objects are written $(\mathcal{P}_\alpha)_{\alpha \in J_{m,n}}$. We recall the main arguments of proofs from [60] and adapt them to our convention to show that these antichains satisfy the conditions of Theorem 3.5.1. This proves Theorem C.

In Section 5 we describe morphisms between the objects $\mathcal{P}_\alpha$ and their shifts. The goal of Section 5 is to prove Theorem F. We call $\mathcal{Y}_{m,n}$ the full subcategory of $D^b(A)$ whose objects are the antichain modules $\mathcal{P}_\alpha$ and their shifts. Knowing that they are intervals and that their corresponding antichains are boolean, Theorem 3.3.4 implies that each hom space is of dimension at most one. Proposition 5.1.12 gives the following canonical factorisation of morphisms in $\mathcal{Y}_{m,n}$ into extensions and degree zero morphisms.

$$\begin{tikzcd} \mathcal{P}_\alpha \arrow[rr] \arrow[dr] & & \mathcal{P}_\beta[|J|] \\ & \mathcal{P}_{q_J(\alpha)}[|J|] \arrow[ur] & \end{tikzcd} \tag{1.7}$$

The extension is explicitly described in Proposition 5.1.8 and further decomposed into *elementary extensions* in Lemma 5.1.13. The proof relies on computations in the homotopy category of complexes as the source of the morphism is identified with its projective resolution and the target is an interval. It follows from Proposition 5.1.12 that the degree one morphisms are parametrised by a subset of the original antichains themselves. It is also crucial to the proof that not all subsets of the antichains yield extensions. Definitions 5.1.5 provides a characterisation of these subsets which we call *allowed* subsets.

The second component of the composition in equation (1.7) is a morphism between intervals of the form $[f(\alpha), \alpha]$. Morphisms between intervals are described by equation (1.5) and comparison of partitions is done term wise. Most of the proofs amount thus to checking inequalities of the form:

$$f(\alpha)_i \leq f(\beta)_i \leq \alpha_i \leq \beta_i$$

for appropriate indices $i \leq m$. Lemma 5.1.1 gives alternative characterisations of morphisms between the objects $\mathcal{P}_\alpha$. Corollary 5.1.3 highlights certain morphisms $\mathcal{P}_\alpha \to \mathcal{P}_{p_J(\alpha)}$ using the new characterisations. Finally, Lemma 5.1.14 provides a decomposition into elementary morphisms.

Next, Proposition 5.2.12 describes the relations between these morphisms. It uses

Lemma 5.2.3 which identifies the relations with the same manipulations of morphisms between intervals and complexes as before. In the process, in Proposition 5.2.2 we interpret the morphisms in a combinatorial setting that links $\mathcal{Y}_{m,n}$ to *Higher Auslander algebras of type A*. In Corollary 5.2.14 we give three slightly different yet significant presentations of the category $\mathcal{Y}_{m,n}$ with generators and relations. This leads us to prove Theorem F. In Proposition 5.3.1 and Lemma 5.3.3 we extract a tilting object from the category $\mathcal{Y}_{m,n}$.

$$T := \bigoplus_{\alpha \in J_{m,n}} \mathcal{P}_\alpha[\kappa_\alpha] \tag{1.8}$$

The integers κ_α ensure that T has no self extensions and are encoded using only the partitions α. The fact that $\mathrm{Thick}(T)$ generates the derived category can already be seen in the proof of 3.5.1 as every projective is obtained as a succession of cones from the family of antichains. One of the presentations of Corollary 5.2.14 concludes the proof of Theorem F while another shows that $\mathrm{End}(T)^{op}$ is isomorphic to the *quadratic dual* $(A_{n+1}^{m-1})^\dagger$ of the higher Auslander algebra of type A. Thus $\mathrm{End}(T)^{op}$ is an intermediate object between the Auslander algebra and its quadratic dual. We conclude this thesis by giving a new characterisation of morphisms in $\mathcal{Y}_{m,n}$ using *interlacing sequences* in a way that mirrors known characterisations of higher Auslander algebras (Proposition 5.4.9).

1.3 Perspectives

We now give a few short term perspectives that arise from the work presented in this thesis. A first step would be to check that the geometric model from [16] has all the correct combinatorial characteristics (see Weights in Section 2.4). Next, one could try to tackle another item from the list of candidate Calabi–Yau posets in [12]. Specifically, can our results, combined with an adapted version of Yıldırım's combinatorics, be used to prove that the lattice of plane partitions is fractionally Calabi–Yau? Other extensions of Yıldırım's combinatorics come to mind when considering that all finite posets can be written as posets of sequences (see Subsection 2.1.3). A topic which was left out of this thesis is the symmetry of the lattice $J_{m,n}$ with regards to m and n and its possible relation to koszulity as indicated through the derived equivalence we establish. Finally, as a general program, what other representation theoretic properties can we detect using antichains? Is it possible to easily detect when an antichain module is an interval? Would that lead us to a general recipe for finding good families of intervals in posets in order to detect the Calabi–Yau property more easily?

Chapter 2

Representation theory of posets

This thesis is about the study of representation theory of the incidence algebras of posets by making use of their combinatorics. In this chapter we recall the results upon which we build the theorems of Chapter 3, 4 and 5. Results are either mentioned because they are used directly or because the author believes they give precious context for subsequent background.

In Subsection 2.1.1, we start by recalling notation and convention of the representation theory of bound quivers. Subsection 2.1.2 contains some results and definitions from Auslander–Reiten theory. We finish this section with basic definitions relating to posets with the aim of discussing poset dimensions. The mathematics of this section were developed no later than the 1970's. Since then, there has been tremendous progress in the different fields we have mentioned. Sections 2.2 and 2.3 record some of these.

In Section 2.2 we recall the construction of the derived category of an abelian category, recall its triangulated structure and discuss important functors. The results of this section date mostly to the 1990's. Some very basic results about triangles will be used in a crucial way in the proof of the main result of Chapter 3.

Section 2.3 gives a short account of our understanding of Higher Auslander theory, which was developed in the 2000's. Our goal here is to introduce higher Auslander algebras of type A as they play an important role in our results.

The author is tempted to say that derived representation theory and Higher Auslander Algebras are two different ways of expanding upon classical representation theory. One could say that the first one arose from the need to replace modules by their better behaved projective resolutions while the second answers natural questions such as *"what if we replace (a hidden) 1 by n?"*. However, higher Auslander–Reiten theory arrived after derived representation theory and lives firmly in that context so the separation is somewhat naive.

Certain aspects of modern representation theory were left completely untouched in this manuscript like the importance of cluster theory, the fact that triangulated categories are being replaced or enhanced by more suited structures such as dg-categories and the rise of the infinity categorical language.

Finally, in Section 2.4 we are able to describe in more details the conjecture of Chapoton which lies at the heart of this thesis.

2.1 Foundations

This section is heavily based on Chapters 1 to 4 of [2] where the reader can find more details on the representation theory of finite dimensional algebras.

2.1.1 Quivers, relations and representations

We first fix some notation and convention[1]. Let $\Bbbk$ be a field. A *quiver* $Q = (Q_0, Q_1, s, t)$ has vertices Q_0, edges Q_1 and the maps $s, t : Q_1 \to Q_0$ assigning respectively a source and a target to each arrow $\alpha \in Q_1$. In this thesis we only consider quivers with a finite number of vertices and arrows. We denote by $\Bbbk Q$ the *path algebra* of the quiver Q. It has basis the finite length paths $\alpha_r \circ \cdots \circ \alpha_1$. where for all $i < r$, $t(\alpha_i) = s(\alpha_{i+1})$. Multiplication is defined by concatenation of paths

$$\beta \circ \alpha = s(\alpha) \xrightarrow{\alpha} [t(\alpha) = s(\beta)] \xrightarrow{\beta} t(\beta).$$

For a given path $p = \alpha_r \circ \cdots \circ \alpha_1$, we set $s(p) = s(\alpha_1)$ to be the source of p and $t(p) = t(\alpha_r)$ to be its target. For each vertex in $i \in Q_0$, denote e_i, the lazy path on i, *i.e.* its associated idempotent. Then, as Q_0 is finite, the unit of $\Bbbk Q$ is $\sum_{i \in Q_0} e_i$. The *radical* of the algebra $\Bbbk Q$ is the ideal generated by paths of length one and denoted R_Q. A two sided ideal I of $\Bbbk Q$ is *admissible* if there exists $m \geq 2$ such that

$$R_Q^m \subseteq I \subseteq R_Q^2$$

If I is an admissible ideal, the pair (Q, I) is a *bound quiver* or quiver with relations and the quotient algebra $\Bbbk Q/I$ the *bound quiver algebra*.

Example 2.1.1. If Q is a finite quiver with no oriented cycles, the zero ideal is admissible. In that case, the quiver algebra is *hereditary*, meaning submodules of projectives are

[1] Like with many convention in mathematics, those chosen in this work are not universal. We apologise in advance to the reader who might be used to arrows going in different directions.

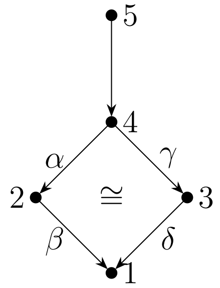


Figure 2.1: A commutative square with one leg

projectives. This is because resolutions of indecomposable modules have length two as elements of the kernel give relations of the algebra.

Example 2.1.2. Figure 2.1 represents a quiver made up of a square with a leg. To make the square commute consider the admissible ideal $I =< \beta \circ \alpha - \delta \circ \gamma >$ generated by a linear combination of paths of length two.

Example 2.1.3. A bound quiver can be associated to any finite poset. Let X be a finite set with a relation $\leq$ which is *transitive*, *reflexive* and *antisymmetric*. A relation $a \leq b$ is a *covering relation* if the *interval* $[a, b] = \{z | a \leq z \text{ and } z \leq b\}$ is of size exactly 2. The Hasse diagram H_Q of the poset is the quiver whose vertices are the elements of the poset and where there is an edge from b to a if and only if $b \geq a$ is a covering relation. Because there is only one way for an element a to be less than an element b, all the paths from b to a should be identified for the poset to be correctly represented by the quiver. Consider the ideal $I = \langle p - q | p, q \text{ paths in } H_q \text{ with } s(p) = s(q) \text{ and } t(p) = t(q)\rangle$. This ideal is admissible because posets have no oriented cycles. The resulting quiver algebra with relations is the *incidence algebra* of the poset. Note that the bound quiver of example 2.1.2 is in fact the bound quiver associated to the poset with five elements $X = \{1, 2, 3, 4, 5\}$ with the order relation defined by $1 \leq 2 \leq 4 \leq 5$ and $1 \leq 3 \leq 4$.

For an admissible ideal I, the condition $R_Q^m \subseteq I$ ensures that any path of length greater than m maps to zero. It follows that the bound quiver algebra is finite dimensional. Meanwhile, the condition $I \subseteq R_Q^2$ ensures that the arrows of the quiver do not map to zero and that the images of the primitive idempotents e_i in $\Bbbk Q/I$ still forms a set of primitive idempotents, their sum being the unit of the algebra. We use the same notation e_i for $i \in Q_0$ to denote the idempotents of $\Bbbk Q/I$ as there will never be a confusion as to which algebra we are referring to. We say a bound quiver algebra is *quadratic* if I is generated by linear combinations of paths of length 2.

A *representation* of a bound quiver $(Q = (Q_0, Q_1), I)$ over a field $\mathbb{k}$ is the datum $((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ where for each $i \in Q_0$, V_i is a $\mathbb{k}$-vector space and $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ is a linear map such that for any element $\sum \lambda_i \cdot \alpha_{i,n_i} \circ \cdots \circ \alpha_{i,1}$ of I we have

$$\sum \lambda_i \cdot f_{\alpha_{i,n_i}} \circ \cdots \circ f_{\alpha_{i,1}} = 0.$$

The dimension vector of the representation Q is the tuple $(\dim(V_i))_{i\in Q_0}$. The *support* of a representation is the set of indices in $i \in Q_0$ such that $V_i \neq 0$. A *morphism* ϕ between two representations $V = ((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ and $U = ((U_i)_{i\in Q_0}, (g_\alpha)_{\alpha\in Q_1})$ is a family of linear transformations $(\phi_i : V_i \to U_i)_{i\in Q_0}$ that make the following square commute for each arrow $\alpha \in Q_1$.

$$\begin{tikzcd}
V_{s(\alpha)} \arrow[r, "f_\alpha"] \arrow[d, "\phi_{s(\alpha)}"] & V_{t(\alpha)} \arrow[d, "\phi_{t(\alpha)}"] \\
U_{s(\alpha)} \arrow[r, "g_\alpha"] & U_{t(a)}
\end{tikzcd}$$

We denote $\mathrm{Rep}_k(Q, I)$ the category of representations of Q with relations I, and $\mathrm{rep}_k(Q, I)$ the full subcategory of $\mathrm{Rep}_k(Q, I)$ whose objects are the quiver representations with finite dimensional vector spaces on each vertex.

Example 2.1.4. The diagrams in Figure 2.2 illustrates representations of the quiver algebra of the commutative square with a leg from example 2.1.2 with the maps

$$\phi : 1 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and ψ the linear form $[1, 1]$. The representations are related by a morphism of representations. The dimension vector of the representation on the left is $(1, 1, 2, 1, 0)$ and the dimension vector of the representation of the right is $(0, 1, 0, 1, 1)$

Theorem 2.1.5. *There is an equivalence between the category $Rep_{\mathbb{k}}(Q, I)$ and the category* $\mathbb{k}Q/I$-Mod *of left modules over the bound quiver algebra. This equivalence restricts to an equivalence between* $\mathrm{rep}_k(Q, I)$ *and* $\mathbb{k}Q/I$-mod *the category of finitely generated $\mathbb{k}Q/I$-modules*

Because $\mathbb{k}Q/I$ has a complete set of primitive idempotents we have the following description for its simple, projective indecomposable and injective indecomposable modules respectively. Let i be an element of Q_0. Its associated simple module S_i has underlying vector space $\mathbb{k}$ with action on a path p defined by $p \cdot 1_{\mathbb{k}} = 0$ unless $p = e_i$ in which case $e_i \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}}$. The projective indecomposable module associated to i is $P_i = \mathbb{k}Q/I \cdot e_i$ and

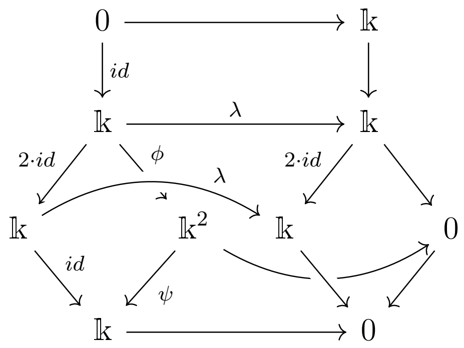


Figure 2.2: A morphism of representations of the commutative square with a leg.

is generated as a vector space by the paths with source i. The injective indecomposable module associated to i is $I_i = (e_i \cdot \Bbbk Q/I)^*$ and is generated as a vector space by the duals of paths with target i.

Example 2.1.6. In example 2.1.4, the representation on the left is isomorphic to $P_4 \oplus S_3$. The representation on the right is isomorphic to I_2.

The importance of bound quivers within the representation theory of finite dimensional algebra is given by the following theorem of Gabriel.

Theorem 2.1.7. *Suppose $\Bbbk$ is an algebraically closed field and let A be a finite dimensional $\Bbbk$-algebra. Then there exists a unique quiver Q and there exists an admissible ideal I such that $A\text{-Mod} \cong Rep_{\Bbbk}(Q, I)$.*

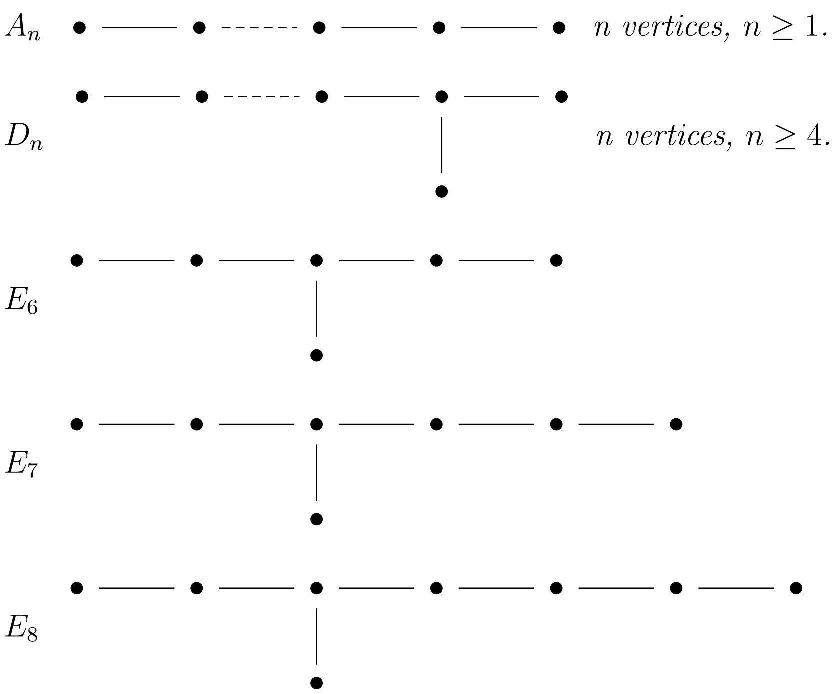
Note that the ideal I need not be unique. This somewhat basic fact will play a surprisingly central role in the last chapter of this thesis. This theorem also illustrates how we are more interested in the *Morita class* of the algebra than in the algebra itself. A classical theorem of Morita characterizes equivalences of categories of modules using special modules.

Theorem 2.1.8 (Morita)**.** *Let A and B be rings. Then the categories of left A and B modules are equivalent if and only if there exists a finitely generated projective A-module P such that there exists a positive integer n for which A is a direct summand of $P^{\oplus n}$ and $B \cong (\operatorname{End}_A P)^{op}$.*

Such a modules P is called *progenerator* as it is projective and generates the entire module category by direct sums and quotients. A Morita equivalence induces an equivalence between the respective categories of finitely generated modules.

The seventies saw the advent of several breakthroughs in the description of the category $Rep_{\Bbbk}(Q, I)$. We distinguish three types of finite dimensional algebras in what is now a famous trichotomy of representation theory. An algebra is of *finite representation type* if it has a finite number of isomorphism classes of finite dimensional indecomposable modules. An algebra is of *tame representation type* if for any given dimension vector, all but a finite number of indecomposable representations are parametrised by a 1-parameter family. An algebra is of *wild representation type* if there is an embedding of the category of finitely generated modules over $\Bbbk\langle x, y\rangle$ in A-Mod where the former is the free algebra on two generators. By a theorem of Drozd, all finite dimensional algebras are of one of these types [19][2]. Hereditary quivers of finite type are classified by the following staple theorem.

Theorem 2.1.9 (Gabriel [24][9])**.** *Let Q be a finite quiver. Then Q is of finite representation type* if and only if *each connected component of its underlying undirected graph is a simply-laced Dynkin diagram as depicted below.*



See [9] for an introductory exposition of the theorem in english.

2.1.2 Auslander–Reiten Theory

Auslander–Reiten theory provides more tools to describe the category of modules of an algebra and detect when an algebra is representation finite. In this subsection we only present the notions and results that will be used in this thesis. We omit central and

[2]with earlier versions in Russian.

interesting aspects of the theory such as almost split sequences and their role in the description of the Auslander–Reiten quiver. The historical source for this section is [5] but we will still refer directly to [2] out of convenience.

Definition 2.1.10 ([2, Definition IV.1.4])**.** Let A be a finite dimensional $\Bbbk$-Algebra. A morphism $f : X \to Y$ in A-mod is said to be *irreducible* provided:

(a) f is neither a *section* nor a *retraction* and

(b) if $f = f_1 \circ f_2$, either f_1 is a retraction or f_2 is a section.

Definition 2.1.11 ([2, Definition A.3.3])**.** The (Jacobson) *radical* of an additive category $\mathcal{C}$ is the two sided ideal $\mathrm{rad}_{\mathcal{C}}$ defined by

$$\mathrm{rad}_{\mathcal{C}}(X,Y) = \{h \in \mathrm{Hom}_{\mathcal{C}}(X,Y) | 1_x - g \circ h \text{ is invertible for any } g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)\} \tag{2.1}$$

Lemma 2.1.12 ([2, Lemma IV.1.6])**.** *Let X, Y be indecomposable modules in A-mod. A morphism $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}_{\ \mathrm{mod}\ A}(X,Y) \setminus \mathrm{rad}^2{}_{\mathrm{mod}\ A}(X,Y)$.*

Construct the *Auslander–Reiten quiver* of a finite dimensional algebra A as follows: take the vertex set Q_0 to be the isomorphism classes of indecomposable A-modules and between two indecomposable modules put as many arrows as the dimensions of the vector space $\mathrm{rad}_{A-\mathrm{mod}}(X,Y)/\,\mathrm{rad}^2_{A-\mathrm{mod}}(X,Y)$. This quiver encodes the structure of the category A-mod as every finite dimensional representation is a direct sum of indecomposable objects and for algebras of finite type, each morphism can be decomposed into irreducible morphisms see [2, Lemma IV.5.6]. Following [2, Definitions I.5.5 and I.5.7] we have the following useful notions.

- A submodule L of M is *superfluous* if for every submodule X of M the equality $L + X = M$ implies $X = M$.
- An epimorphism $h : M \to N$ is *minimal* if $\mathrm{Ker}\, h$ is superfluous in M. An epimorphism $h : P \to M$ is a *projective cover* of M if P is a projective module and h is a minimal epimorphism.
- An exact sequence

$$\cdots \to P_m \xrightarrow{h_m} P_{m-1} \to \cdots \to P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \to 0$$

in A-mod is a *minimal projective resolution* of M if for each $j \geq 0$, the morphism $h_j : P_j \to \mathrm{Im}\, h_j$ is a projective cover. The truncated sequence $P_1 \to P_0 \to M$ is called a *minimal projective presentation* of M.

Example 2.1.13. In a finite quiver $Q = (Q_0, Q_1)$ with no oriented cycles, the projective cover of the simple S_i is the projective module P_i for any $i \in Q_0$.

It is well known that every finitely generated module over a finite dimensional algebra admits a projective cover. It follows that every module has a minimal projective resolution. See [2, Theorem 5.8] for instance. Every minimal projective resolution of a module M has the same length which we call the *projective dimension* of M and denote it $\operatorname{pdim} M$. For an algebra A, the *global dimension* is the supremum of the projective dimension ranging over all left A modules. It is denoted $\operatorname{gldim} A$ and can be infinite. In this thesis we consistently work with projective resolutions, however we point out that dual notions exists with injective modules.

- A monomorphism $u : M \to E$ is *minimal* if every non zero submodule X of E has a non zero intersection with $\operatorname{Im} u$.
- A monomorphism $u : M \to I$ is an *injective hull* if I is an injective module and u is minimal.
- An exact sequence written with cohomological conventions

$$0 \to M \xrightarrow{u^0} I^0 \xrightarrow{i^1} I^1 \to \cdots \to I^{m+1} \xrightarrow{i^m} I^m \to \ldots$$

  in A-mod is a *minimal injective resolution* if for each $m \geq 0$, the embedding $\operatorname{Im}(u^m) \hookrightarrow I^m$ is a minimal monomorphism. The truncated sequence $M \to I^0 \to I^1$ is called a *minimal injective presentation* of M.

In the category of finitely generated modules over a finite dimensional algebra, every module has a minimal injective resolution.

Example 2.1.14. In a finite quiver $Q = (Q_0, Q_1)$ with no oriented cycles, the injective hull of the simple S_i is the injective module I_i for any $i \in Q_0$.

The length of a minimal injective resolution of a modules M is the *injective dimension* of the module denoted $\operatorname{idim} M$.

Theorem 2.1.15 ([44])**.** *The maximum of the injective dimension ranging over all left A-modules coincides with the global dimension.*

Theorem 2.1.16 (Auslander [44, Theorem 5.73])**.** *Let A be finite dimensional algebra with Jacobson radical $J = \operatorname{rad} A$. Let $\{S_i\}$ be a complete set of simple left A-modules (up to isomorphisms). Then*

$$\operatorname{gldim} A = \max(\operatorname{pdim}(S_i)) = \operatorname{pdim}(A/J).$$

This theorem holds for a larger class of rings but will not be used in its more general form. The global dimension is one measure of the complexity of the category of modules over an algebra. The higher it is the more layers of projective modules one needs in order to describe finitely generated modules. Other statistics behave differently. Consider A as a left A module and let

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^m \to \ldots$$

be a minimal injective resolution of A. As in [5] we define the *dominant dimension* of A to be the maximal integer i (or ∞) such that for all $j < i$, the module I^j is a projective-injective module. We denote it $\operatorname{domDim}(A)$.

Example 2.1.17. The square with a leg 2.1 has a unique projective-injective module $P_5 \cong I_1$ and its incidence algebra $A_\square$ has the following minimal injective resolution.

$$0 \to {}_AA \to I_1^{\oplus 5} \to I_3^{\oplus 2} \oplus I_2^{\oplus 2} \oplus I_5 \to I_4 \to 0$$

Hence its dominant dimension is 1.

Proposition 2.1.18. *A minimal projective resolution is a projective resolution of minimal length on M.*

Definition 2.1.19. An *Auslander algebra* is an algebra A whose global and dominant dimensions satisfy the two inequalities:

$$\operatorname{gldim}(A) \leq 2 \leq \operatorname{domDim}(A).$$

We now state a famous theorem of the representation theory algebras, linking the notions of finiteness we have introduced so far in a surprising way.

Theorem 2.1.20 (Auslander correspondance [3])**.** *There is a one to one correspondence between Morita equivalence classes of representation finite, finite dimensional algebras and Morita equivalence classes of finite dimensional Auslander algebras. It is given by*

$$\Lambda \mapsto \Gamma := \operatorname{End}(M)$$

where M is an additive generator of Λ-mod.

Example 2.1.21. Consider the quiver algebra of equioriented type A on 5 vertices. Its quiver is also the Hasse diagram of a total order on 5 elements. It is known that the

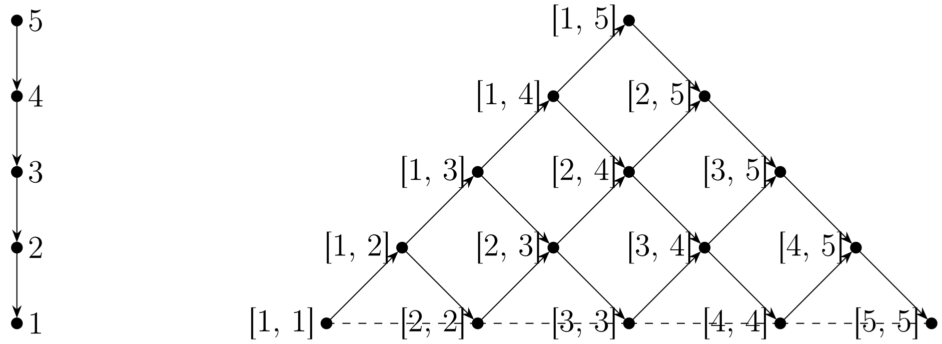


Figure 2.3: The quiver A_5 and the Auslander–Reiten quiver of its path algebra

indecomposable A_5-modules are the intervals of the poset. The morphisms between them follow the rule from equation (1.5). In Figure 2.3, dashed lines indicate zero relations and squares commute. The labels indicate the intervals.

Noteworthy functors Fix a finite dimensional algebra with finite global dimension. In this thesis, we generally work with left modules. In this paragraph we consider the categories of finitely generated left but also right modules over A. The latter is denoted $\operatorname{mod} A$. There exist several functors that link left and right A modules.

$$D : A\text{-mod} \to \operatorname{mod} A \qquad\qquad (-)^t : A\text{-mod} \to \operatorname{mod} A$$
$$M \mapsto \operatorname{Hom}_k(M, k) \qquad\qquad M \mapsto \operatorname{Hom}_A(M, A)$$

With the functor D, also called *standard duality*, we make more precise the duality between projective modules and injective modules [2, Theorem 5.13]: D is a contravariant equivalence of abelian categories that sends the projective (*resp.* injective) left modules to the injective (*resp.* projective) right modules, left projective covers (*resp.* injective envelope) to right injective envelopes (*resp.* projective covers). By composing these functors above together we get other interesting functors. The *Nakayama functor* is the endofunctor of mod-A obtained by the following composition

$$\nu : A\text{-mod} \to A\text{-mod}$$
$$M \mapsto D\operatorname{Hom}_A(M, A).$$

It is ubiquitous in representation theory. There is an isomorphism of functors

$$\nu \cong DA \bigotimes_A ? \tag{2.2}$$

as both functors are right exact and coincide on projective modules [2, Lemma III.2.8]. The Nakayama functor is an equivalence between the subcategory of projective modules and the subcategory of injective modules [2, Proposition III.2.10]. As an indication of the proof notice that left module morphisms from the indecomposable projective module e_xA to A are entirely determined by the image of the idempotent e_x. Hence we have an isomorphism $\mathrm{Hom}_A(e_xA, A) \simeq A \cdot e_x$ which leads to $\nu P_x = D\,\mathrm{Hom}_A(e_xA, A) \simeq I_x$. The functor $\nu^{-1} = \mathrm{Hom}_A(DA, ?)$, while being well defined on all left modules, acts as a quasi inverse to the Nakayama functor on the subcategory of injective left modules. The following example illustrates how the Nakayama functor is not an equivalence of categories in general.

Example 2.1.22. If P is a poset with at least two elements and a maximum $\hat{1}$, then one can check that $\nu S_{\hat{1}} \simeq 0$ using the characterization of morphisms between intervals of posets given in equation (1.5).

One can also consider the *transpose* $Tr(M)$ of a left A-module M. Let $P_1 \xrightarrow{p_1} P_0 \to M \to 0$ be a minimal projective presentation of M. When we apply the functor $(-)^t$ to this sequence we get a sequence of right modules which can be completed into a short exact sequence $0 \to M^t \to P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{Coker}(p_1^t) \to 0$. We set $Tr(M) = \mathrm{Coker}(p_1^t)$. The following Proposition describe how the transpose is a bijection on certain isomorphism classes of indecomposable modules.

Proposition 2.1.23 ([2, Proposition 2.1])**.** *Let M be an indecomposable module. M is projective if and only if $\mathrm{Tr}(M) = 0$. If M is not projective then $\mathrm{Tr}\,M$ is indecomposable and $\mathrm{Tr}(\mathrm{Tr}(M)) \cong M$. Moreover, if N is also an indecomposable non projective module, then $M \cong N$ if and only if $\mathrm{Tr}(M) \cong \mathrm{Tr}(N)$.*

Consider the procedure $M \mapsto D\,\mathrm{Tr}(M)$. It is well defined but not functorial. It sends to zero the projective modules but acts bijectively on classes of non projective indecomposable modules. To make the translation into an automorphism of categories, we construct the *projective stable category* $\underline{\mathrm{mod}}$-A by taking the quotient of mod-A by the ideal generated by morphisms which factor through projectives [2, Proposition 2.2]. Similarly, we can construct the *injective stable category* $\overline{\mathrm{mod}}$-A. Define the *Auslander–Reiten translation* and its inverse as follows:

$$\begin{aligned} \tau : A\text{-}\underline{\mathrm{mod}} &\to A\text{-}\overline{\mathrm{mod}} \\ M &\mapsto D\,\mathrm{Tr}(M) \end{aligned} \qquad\qquad \begin{aligned} \tau^{-1} : A\text{-}\overline{\mathrm{mod}} &\to A\text{-}\underline{\mathrm{mod}} \\ M &\mapsto \mathrm{Tr}\,D(M). \end{aligned}$$

When looking at their action on modules and not on morphisms, the Nakayama functor and the Auslander–Reiten translations are both related to very basic dualities and act as bijections on different parts of the category of modules. The following proposition gives the precise way in which these functors are linked.

Proposition 2.1.24. *(a) Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of an A-module M. Then there exists an exact sequence*

$$0 \to \tau M \to \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \to 0.$$

(b) Let $0 \to N \xrightarrow{i_0} E_0 \xrightarrow{i_1} E_1$ be a minimal injective presentation of an A-module M. Then there exists an exact sequence

$$0 \to \nu^{-1} N \xrightarrow{\nu^{-1} i_0} \nu^{-1} E_0 \xrightarrow{\nu^{-1} i_1} \nu^{-1} E_1 \to \tau^{-1} M \to 0.$$

We finish this subsection with a small amount of context regarding the Auslander–Reiten translation as it plays a central role in describing the category of finitely generated modules over finite dimensional algebras. The main ideas are summed up in the following theorem. We leave certain notions, written in red, undefined, as we do not use them. We point to [2, Chapter IV] for more details and to [4] for an historical source.

Theorem 2.1.25 ([2, Theorem IV.3.1, Lemma IV.4.8])**.** *(a) For any indecomposable non projective finitely generated A-module M there exists an almost split sequence*

$$0 \to \tau M \to E \to M \to 0.$$

(b) For any indecomposable non injective finitely generated A-module N, there exists an almost split sequence

$$0 \to N \to F \to \tau^{-1} N \to 0.$$

This makes the Auslander–Reiten quiver into a translation quiver.

This theorem gives rise to a procedure, the *knitting algorithm* which can be used to describe the entire category of finitely generated modules when the algebra is representation finite [1]. See Figure (2.4) where the dashed arrows indicated the action of the Auslander–Reiten translation when it is non zero.

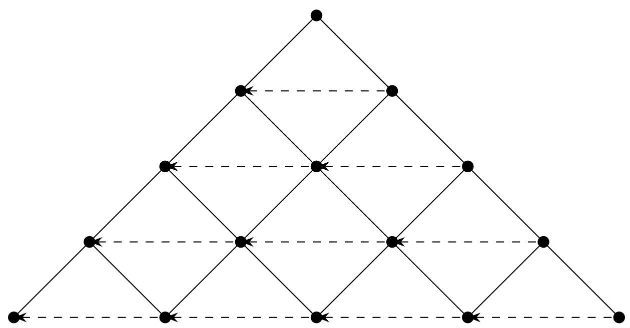


Figure 2.4: The Auslander–Reiten translation of A_5

2.1.3 Posets

In this subsection we discuss combinatorial properties of posets. These concepts will be used in representation theoretic contexts to cary out homological computations that are in general hard. Fix a finite poset $(X, \leq)$. A *linear extension* of the order $\leq$ on X is a total ordering on X which refines the partial order $\leq$. A poset has many statistics characterizing its complexity. For instance the size of the largest antichain in X is called the *width*. A chain has width 1. We denote the width of X by $w(X)$. Another statistic would be the maximal integer k such that there exists $x \in X$ which covers exactly k elements. We call this the *maximal covering number* and denote it $\mathrm{cov}(X)$. Recent work in algebraic combinatorics points to the maximal covering number having interesting algebraic interpretations [6]. The following statistic is more complex to define but is more commonly used in combinatorics.

Definition 2.1.26. (Dushnik-Miller order dimension [20]) The order dimension of a poset is the minimal integer t such that there exists t linear extensions of $\leq$ whose intersection is equal to $\leq$.

Theorem 2.1.27 ([40])**.** *Let t be the order dimension of X. Then t is the minimal integer such that X embeds as a poset in the product of chains $\mathbb{R}^t$ equipped with term-wise comparison*

Proof. We write the finite poset X as $\{1, \ldots, n\}$. Let $\mathcal{R}_1, \ldots, \mathcal{R}_t$ be linear extensions of $\leq$ satisfying $\leq = \cap_i \mathcal{R}_i$. The data of an ordering $\mathcal{R}_i$ on X corresponds to a permutation σ_i on X such that

$$\sigma_i(1)\mathcal{R}_i\sigma_i(2)\mathcal{R}_i \ldots \mathcal{R}_i\sigma_i(n).$$

The following map is an order preserving embedding

$$\begin{aligned} P &\to \quad \mathbb{R}^t \\ x &\mapsto (\sigma_1(x), \ldots, \sigma_t(x)). \end{aligned}$$

This shows that the least integer for which such an embedding exists is at most t. Conversely, consider an order preserving embedding $\iota : P \to \mathbb{R}^s$. We can assume that the projection π_i on the i^{th} copy of $\mathbb{R}$ in $\mathbb{R}^s$ is injective. Then the permutation

$$\sigma_i := (\pi_i \circ \iota(1), \ldots, \pi_i \circ \iota(n))$$

defines a total order $\mathcal{R}_i$ on $X = \{1, \ldots, n\}$. One can check that $\leq = \cap_i \mathcal{R}_i$. □

The posets that we consider in this thesis are often *lattices*, meaning that pairs of elements $x, y \in X$ have a well defined

- least upper bound, denoted $x \vee y$ (read "x *join* y");
- greatest lower bound, denoted $x \wedge y$ (read "x *meet* y");

in which case the binary operations $\vee$ and $\wedge$ are compatible with the partial order *i.e.*, if $x_1 \leq x_2$ and $y_1 \leq y_2$ then $x_1 \vee y_1 \leq x_2 \vee y_2$ and $x_1 \wedge y_1 \leq x_2 \wedge y_2$. Equivalently, a lattice can be defined as an algebraic structure on the set X using only the join and meet binary operations together with absorption axioms. In that case, an order relation on the set X can be defined in order to recover the previous characterization. Every finite lattice is complete, implying it has a *least* and *maximal* element denoted respectively $\hat{0}$ and $\hat{1}$. A lattice is said *distributive* if the join distributes over the meet and vice versa *i.e.* for $x, y, z \in X$, we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Example 2.1.28. Let m and n be positive integers. Consider the set of non decreasing sequences of length m with values in $[\![0, n]\!]$ ordered with termwise comparison. Denote this poset $J_{m,n}$. It is a lattice. For elements $(a_i)_{i\leq m}$ and $(b_i)_{i\leq m}$ in $J_{m,n}$ compute the join and the meet by taking the join and meet termwise as follows

$$(a_i)_{i\leq m} \vee (b_i)_{i\leq m} = (\max(a_i, b_i))_{i\leq m} \text{ and } (a_i)_{i\leq m} \wedge (b_i)_{i\leq m} = (\min(a_i, b_i))_{i\leq m}.$$

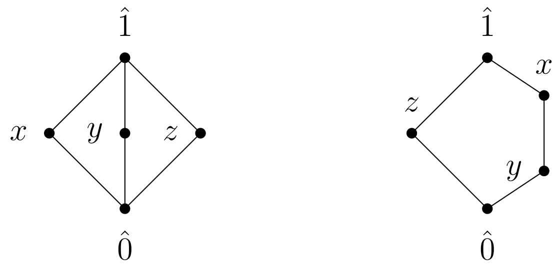


Figure 2.5: The diamond and the pentagon lattices

Moreover, we can check that it is distributive with a direct computation

$$\begin{aligned}(a_i)_{i\leq m} \wedge [(b_i)_{i\leq m} \vee (c_i)_{i\leq m}] &= (\min(a_i, \max(b_i, c_i)))_{i\leq m}\\ &= (\max(\min(a_i, b_i), \min(a_i, c_i)))_{i\leq m}\\ &= [(a_i)_{i\leq m} \wedge (b_i)_{i\leq m}] \vee [(a_i)_{i\leq m} \wedge (c_i)_{i\leq m}].\end{aligned}$$

The computation is similar for the second distributivity identity.

A sublattice is a subset of X stable under the meet and the join operation. There are combinatorial criteria to identify distributive lattices. The following is well known and attributed to Garrett Birkhoff.

Theorem 2.1.29. *A lattice is distributive if and only if it does not contain sublattices isomorphic to the diamond or the pentagon lattices (Figure 2.5).*

While the notion of distributive lattices is very useful, it can be both strengthened and weakened. On the stronger side, a distributive lattice is *boolean* if every element has a *complement* *i.e.* for all $x \in X$ there exists $x' \in X$ such that $x \vee x' = \hat{1}$ and $x \wedge x' = \hat{0}$. We call *atoms* the elements that cover $\hat{0}$. The following theorem is due to Stone [55]

Theorem 2.1.30. *A finite lattice is boolean if and only if it is isomorphic to the power set of the atoms of X.*

In particular, the meet, join and complement operation of boolean lattices correspond to the intersection, union and complement operations of power sets and its cardinal is always a power of two.

A similar *representation* result exists for distributive lattices using a more elaborate set of subsets of the lattice. An element x of X is *join irreducible* if it covers exactly one element of X *i.e.* for all a, b such that $a \vee b = x$, we have $a = x$ or $b = x$. Join irreducible elements are to lattices what irreducible elements are to integral domains. An *order ideal* is a subset I of X that is downward closed *i.e.* if $x \in I$ and $y \leq x$ then $y \in I$. The following famous theorem is due to Birkhoff.

Theorem 2.1.31 (Fundamental theorem of finite distributive lattices [7])**.** *A finite lattice is distributive if and only if it is isomorphic to the lattice of order ideals of its poset of join irreducible elements.*

Distributive lattices are to lattices what unique factorisation domains are to domains.

Example 2.1.32. Let n be a positive integer. The set of divisors of n can be equipped with the order relation

$$a \leq b \Leftrightarrow a|b.$$

This makes it a distributive lattice where the meet is given by the gcd of two element and the join is their lcm. The join irreducible elements of this lattice are the prime powers dividing n.

Example 2.1.33. Recall from example 2.1.28 the lattice $J_{m,n}$ of non decreasing sequences of length m in $[\![0, n]\!]$. To characterize its join irreducible elements we notice that the covering relations of this lattice correspond to relations

$$a = (a_1, \dots, a_i, \dots, a_m) > (a_1, \dots, a_i - 1, \dots, a_m).$$

For the second sequence to be non decreasing, it must be that $a_{i-1} < a_i$. Hence the element a of $J_{m,n}$ covers exactly one other element of the lattice if and only if it has exactly one non zero value. In that case a can be written as $(0, \dots, 0, j, \dots, j)$ with i copies of the value $j \in [\![0, n]\!]$ and $m - i$ copies of the value 0. We denote these sequences $(0^{m-i}, j^i)$. Hence there is an increasing bijection between the join irreducible $\mathrm{Irr}(J_{m,n})$ and the product of two chains $A_m \times A_n$.

$$\begin{aligned} \mathrm{Irr}(J_{m,n}) &\to A_m \times A_n \\ (0^{m-i}, j^i) &\mapsto (i, j) \end{aligned}$$

To illustrate Birkhoff's theorem, note that an order ideal I of $A_m \times A_n$ can be drawn as a path in an $m \times n$ grid as on the right side of Figure 2.6. The elements of the order ideal are the points of the grid which lie below the path in the picture. Because I is closed downward, counting the number of points in each column that belong to I, with increasing first value, gives a monotone sequence which completely determines the ideal. For the path in the picture, the corresponding sequence is $(2, 3)$ indicated by a black

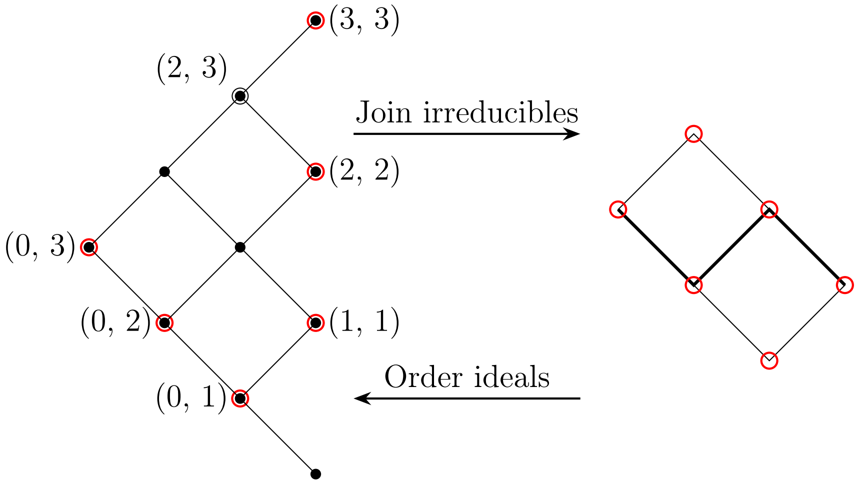


Figure 2.6: $J_{2,3}$ and its poset of join irreducible elements

circled dot on the left. With this example we recover the first definition we gave of $J_{m,n}$ in the introduction of this thesis: it is the lattice of order ideals of an $m \times n$ grid.

Theorem 2.1.34 (Dilworth [18]). *Let L be a finite distributive lattice. Then*

$$\mathrm{OrdDim}(L) = \mathrm{cov}(L).$$

Example 2.1.35. Because of the isomorphism $J_{m,n} \cong J_{n,m}$ we can assume that $m \leq n$. The definition of distributive lattice $J_{m,n}$ already took the form of an embedding into $\mathbb{R}^m$ giving a clear upper bound to the order dimension of X. The elements covering the largest amount of other elements are the ones which correspond to increasing sequences which do not start with zero. They cover exactly m other elements. This shows that in general we have $\mathrm{OrdDim}(J_{m,n}) = \mathrm{cov}(J_{m,n}) = \min(m, n)$.

The theorem does not in general hold for lattices that are not distributive.

Example 2.1.36. The diamond lattice (left, Figure 2.5) has maximal covering number 3 but order dimension at most two. To compute this upper bound for the order dimension notice that the partial order relation on this poset is the intersection of the two following total order relations: $\{\hat{0} \leq x \leq y \leq z \leq \hat{1}\}$ and $\{\hat{0} \leq z \leq y \leq x \leq \hat{1}\}$.

We now turn to properties that are weaker than distributivity. We say that a lattice is *join semidistributive* if whenever $x, y, z \in X$ satisfy $x \vee y = x \vee z$ then $x \vee (y \wedge z) = x \vee y$. Dually, a lattice is *meet semidistributive* if whenever $x, y, z \in X$ satisfy $x \wedge y = x \wedge z$ then $x \wedge (y \vee z) = x \wedge y$. We say that a lattice is *semidistributive* if it is both join and meet semidistributive.

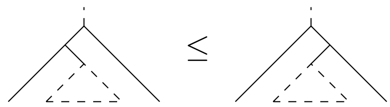


Figure 2.7: Tree rotation

Example 2.1.37. The Tamari lattices form a famous infinite family of semidistributive lattices. In this thesis, we choose to define Tamari n as the set of binary trees with $n+1$ leaves and order generated by tree rotations as depicted in Figure 2.7. Tamari 3 is the pentagon on the right of Figure 2.5. For a proof that the Tamari lattice is semidistributive see [57] or [26].

Example 2.1.38. The diamond lattice on the left of Figure 2.5 is not join semidistributive as $x \vee y = \hat{1} = x \vee z$ while $x \vee (y \wedge z) = x \neq x \vee y$. A similar calculation shows that it is not meet semidistributive.

Recently, a representation theorem in the spirit of the Birkhoff's fundamental theorem for distributive lattices was proved for semidistributive lattices [48]. Semidistributive lattices also have a pattern avoidance characterization extending Theorem 2.1.29 [38].

2.2 Derived representation theory

This section is taken from [61]. Here is a naive motivation to the following section. We construct the homotopy category and the derived category because we want to replace complexes by projective resolutions. In doing so we lose the abelian structure so we introduce triangulated categories.

2.2.1 The derived category

Let $\mathcal{A}$ be an additive category. A *complex* is a sequence of objects $C = (C_i)_{i\in\mathbb{Z}}$ of $\mathcal{A}$ along with a sequence of morphisms $(\partial_i)_{i\in\mathbb{Z}}$ with $\partial_i : C_i \to C_{i-1}$ such that $\partial_i \circ \partial_{i+1} = 0$. Morphisms of complexes are sequences $(f_i : C_i \to D_i)_i$ such that $f_i \circ \partial^C_{i+1} = \partial^D_{i+1} \circ f_{i+1}$. This gives a category $C(\mathcal{A})$. The *shift functor* sends a complex $C = ((C_i)_i, (\partial_i)_i)$ to a complex $C' = ((C'_i)_i, (\partial'_i)_i)$ where

$$C'_i = C_{i-1} \text{ and } \partial'_i = -\partial_{i-1}$$

similarly shifting morphisms of complexes. We write it as [1], its adjoint as [-1] and the successive compositions of either of them $[n]$ for any integer n. The *homotopy relation* is defined on maps but helps us compare objects as well. Let $f, g : C \to D$ be morphisms between two chain complexes. We say that f and g are homotopic if there exists a map h of degree 1 such that

$$\partial^D_{n+1} \circ h_n + h_{n-1} \circ \partial^C_n = f_n - g_n.$$

It is standard to write $f \sim g$ when g is homotopic to f and it is easy to check that it is an equivalence relation. A map $f : C \to D$ is a homotopy equivalence if there exists $g : D \to C$ such that

$$f \circ g \sim id_D \text{ and } g \circ f \sim id_C$$

The *homotopy category* $\mathrm{Ho}(\mathcal{A})$ of $\mathcal{A}$ is defined to have the same objects as $C(\mathcal{A})$ and morphisms

$$\mathrm{Hom}_{Ho(\mathcal{A})}(C, D) := \mathrm{Hom}_{C(\mathcal{A})}(C, D)/\sim .$$

Composition of maps is inherited by the composition of the category of complexes. There is an essentially surjective functor from the category of complexes to the homotopy category, sending objects to themselves and maps to their equivalence class. The homotopy category, together with this functor satisfies a universal property. In addition to that, the embedding of $\mathcal{A}$ in $C(\mathcal{A})$, as complexes concentrated in degree zero, extends to the homotopy category and is full and faithful. If the category $\mathcal{A}$ is in fact abelian we denote the *homology of degree i* of a complex $C = ((C_i)_i, (\partial_i)_i)$ by $\mathrm{H}_n(C) = \mathrm{Ker}(\partial_i)/\,\mathrm{Im}(\partial_{i+1})$. For any map of chain complexes $f : C \to D$ we have a well defined map on homology:

$$\begin{aligned} \mathrm{H}_n(C) &\to \mathrm{H}_n(D) \\ [x] &\mapsto [f_n(x)]. \end{aligned}$$

It is easy to check that H_n is a functor from $C(\mathcal{A})$ to the category of abelian groups. Two complexes are quasi-isomorphic if there exists a map $f : C \to D$ such that for all $i \in \mathbb{Z}$, the map $\mathrm{H}_i(f)$ is an isomorphism. The morphism f is then called a *quasi-isomorphism*. Let $f : C \to D$ be a morphism of chain complexes. Then the *mapping cone* of f is the complex whose objects are $\mathrm{Cone}(f)_n = C_{n-1} \oplus D_n$ with connecting maps

$$C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -\partial^C_{n-1} & 0 \\ f_{n-1} & \partial^D_n \end{pmatrix}} C_{n-2} \oplus D_{n-1}$$

The mapping cone is a neat construction that can help us identify quasi-isomorphisms as well as homotopy equivalences. It also plays an important role in the triangulated structure of the derived category which we will define shortly.

Fact 2.2.1. *A map $f : C \to D$ is a quasi-isomorphism* if and only if $\operatorname{Cone}(f)$ *is exact. The map f is a homotopy equivalence* if and only if $\operatorname{Cone}(f)$ *is zero homotopic.*

We give a quick construction of the *derived category* of the abelian category $\mathcal{A}$. We omit set theoretic considerations and simply point out that when $\mathcal{A}$ is the category of modules over an algebra, quasi-isomorphisms form a locally small multiplicative system [59, Remark 10.3.6], [25, Chapter 1]. The derived category of $\mathcal{A}$ is written $D(\mathcal{A})$, has for objects the complexes of objects in $\mathcal{A}$. For its morphisms consider the triples $\phi : X \xleftarrow{\nu} Z \xrightarrow{\alpha} Y$ where ν is a quasi isomorphism and α is a homotopy class of morphism of complexes, for any pair of complexes X and Y. We say that $\psi : X \xleftarrow{\nu'} Z' \xrightarrow{\alpha'} Y$ covers ϕ if there exists a map γ making the following diagram commute

$$\begin{tikzcd} & Z' \arrow[dl, "\nu'"'] \arrow[dr, "\alpha'"] \arrow[dd, "\gamma"] & \\ X & & Y \\ & Z \arrow[ul, "\nu"] \arrow[ur, "\alpha"'] & \end{tikzcd} .$$

We say that two triples are equivalent if they are both covered by the same triple. The vector space $\operatorname{Hom}_{D(\mathcal{A})}(X, Y)$ is obtained by quotienting the space of triples by this relation which is indeed an equivalence relation.

In the category of complexes we can consider the full subcategories of complexes bounded on the left, bounded on the right or just bounded which we write respectively $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ and $C^b(\mathcal{A})$. Their essential image in the homotopy category are written $\operatorname{Ho}^+(\mathcal{A})$, $\operatorname{Ho}^-(\mathcal{A})$ and $\operatorname{Ho}^b(\mathcal{A})$ and in the derived category $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$. We will also consider the subcategory $C^{-,b}(\mathcal{A})$ of complexes bounded on the right with bounded homology and its essential image in the homotopy category $\operatorname{Ho}^{-,b}(\mathcal{A})$. The derived categories together with the functor

$$\begin{aligned} u : \operatorname{Ho}(\mathcal{A}) &\to D(\mathcal{A}) \\ (C_i)_i &\mapsto (C_i)_i \\ (f : X \to Y) &\mapsto (X \xrightarrow{id_X} X \xleftarrow{f} Y) \end{aligned} \tag{2.3}$$

satisfies a universal property. This functor restricts to the subcategories enumerated above. Moreover the functor $\mathcal{A} \to D(\mathcal{A})$ sending modules to complexes concentrated in degree zero is fully faithful.

When A is a finite dimensional $\Bbbk$-algebra, we can consider different base categories related to finiteness conditions: the category of left modules over A, denoted A-Mod, the category of finitely generated left modules, A-mod, the category of projective modules A-Proj, the category of finitely generated projective modules A-proj, *etc.* The categories of projective modules and of finitely generated modules play an important technical role in the derived representation theory of finite dimensional algebras which we will now describe succinctly. We call *projective resolution of a complex* C a complex of projective modules P with a quasi-isomorphism $P \to C$. It is easy to show that projective resolution of modules are projective resolutions of complexes if we see the module as a complex concentrated in degree zero. Just like for modules we would like every object of the bounded derived category to have a projective resolution. We have the following result

Proposition 2.2.2. *Let A be an algebra and X an object in $D^-(A\text{-Mod})$. Then X has a projective resolution.*

They are useful in part because they make some hom spaces easier to compute. Using cones to characterize quasi-isomorphisms and homotopy equivalences we can show that a complex of projectives is exact if and only if it is zero homotopic and deduce the following proposition.

Proposition 2.2.3 ([54, Tag 064B])**.** $\mathrm{Hom}_{\mathrm{Ho}}(P, ?) \cong \mathrm{Hom}_D(P, ?)$

Combining Propositions 2.2.2 and 2.2.3 we have the following equivalences of categories

Theorem 2.2.4 ([61, Proposition 3.5.43])**.** *Let A be an algebra. Then the functor from equation (2.3) induces equivalences*

- $D^-(A\text{-Mod}) \cong \mathrm{Ho}^-(A\text{-Proj})$
- $D^b(A\text{-Mod}) \cong \mathrm{Ho}^{-,b}(A\text{-Proj})$

and if A is noetherian over a commutative ring

- $D^-(A\text{-mod}) \cong \mathrm{Ho}^-(A\text{-proj})$
- $D^b(A\text{-mod}) \cong \mathrm{Ho}^{-,b}(A\text{-proj})$

In each case, the adjoint is called the projective resolution functor*.*

Dual results hold for injective resolutions.

2.2.2 Triangulated structure

In the previous section we fixed an abelian category $\mathcal{A}$. What structures do the category of complexes, the homotopy category and the derived category have? While $C(\mathcal{A})$ inherits the abelian structure of $\mathcal{A}$ by computing kernels and cokernels degree by degree, this does not apply to $\mathrm{Ho}(\mathcal{A})$ and $D(\mathcal{A})$. They are instead triangulated. In this subsection we recall the definition of a triangulated category as well as the properties we will use in the thesis before spelling out the triangulated structure of the homotopy and derived category of an abelian category $\mathcal{A}$.

Let $\mathcal{C}$ be an additive category and let T be an additive self-equivalence of $\mathcal{C}$. A *triangle* is the data of three objects X, Y and Z along with morphisms

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} TX.$$

A *morphism of triangles* is a commutative diagram of the form :

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\xi"] & Y \arrow[r, "\beta"] \arrow[d, "\eta"] & Z \arrow[r, "\gamma"] \arrow[d, "\zeta"] & TX \arrow[d, "T\xi"] \\ X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & TX' \end{tikzcd}$$

A morphism of triangles is an isomorphism if the maps ξ, η and ζ are isomorphisms in $\mathcal{C}$. An additive category $\mathcal{C}$ along with a self equivalence T and a class of triangles called *distinguished* is a *triangulated category* if it satisfies the following axioms.

Tr1 Any triangle isomorphic to a distinguished triangle is distinguished as well. For all object X, the triangle $X \xrightarrow{id_X} X \to 0 \to TX$ is distinguished. Every morphism $X \xrightarrow{\alpha} Y$ can be completed into a distinguished triangle called triangle *above* α.

Tr2 If $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} TX$ is distinguished then so are $Y \xrightarrow{\beta} Z \xrightarrow{\zeta} TX \xrightarrow{-T\alpha} TY$ and $T^{-1}Z \xrightarrow{-T^{-1}\zeta} X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$.

Tr3 Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} TX$ and $X' \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} Z' \xrightarrow{\gamma'} TX'$ be two distinguished triangles. If there exist maps ξ and η such that $\eta \circ \alpha = \alpha' \circ \xi$ then there exists $\zeta : Z \to Z'$ completing the following morphism of triangles.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\xi"] & Y \arrow[r, "\beta"] \arrow[d, "\eta"] & Z \arrow[r, "\gamma"] \arrow[d, "\exists", swap] \arrow[d, "\zeta"] & TX \arrow[d, "T\xi"] \\ X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & TX' \end{tikzcd}$$

Tr4 Given $X_1 \xrightarrow{\alpha_3} X_2 \xrightarrow{\alpha_1} X_3$, write $\alpha_2 = \alpha_1 \circ \alpha_3$.With these 3 morphisms we can form the following three triangles using **TR1**

$$X_1 \to X_2 \xrightarrow{\beta_3} Z_3 \xrightarrow{\gamma_3} TX_1$$
$$X_1 \to X_3 \xrightarrow{\beta_2} Z_2 \xrightarrow{\gamma_2} TX_1$$
$$X_2 \to X_3 \xrightarrow{\beta_1} Z_1 \xrightarrow{\gamma_1} TX_2$$

which are connected by a certain number of morphisms in $\mathcal{C}$. Axiom **TR4** states that this diagram can be completed into the following commutative diagram in which only δ_1, δ_2 and δ_3 are new morphisms. They form a distinguished triangle.

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha_3"] \arrow[d, "id"] & X_2 \arrow[r, "\beta_3"] \arrow[d, "\alpha_1"] & Z_3 \arrow[r, "\gamma_3"] \arrow[d, "\delta_1"] & TX_1 \arrow[d, "id"] \\
X_1 \arrow[r, "\alpha_2"] \arrow[d, "\alpha_3"] & X_3 \arrow[r, "\beta_2"] \arrow[d, "id"] & Z_2 \arrow[r, "\gamma_2"] \arrow[d, "\delta_2"] & TX_1 \arrow[d, "T(\alpha_3)"] \\
X_2 \arrow[r, "\alpha_1"] & X_3 \arrow[r, "\beta_1"] & Z_1 \arrow[r, "\gamma_1"] \arrow[d, "\delta_3"] & TX_2 \arrow[d, "T(\beta_3)"] \\
 & & TZ_3 \arrow[r, "id"] & TZ_3
\end{tikzcd}$$

Lemma 2.2.5. *Let $\mathcal{C}$ be a triangulated category and let*

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\xi"] & Y \arrow[r, "\beta"] \arrow[d, "\eta"] & Z \arrow[r, "\gamma"] \arrow[d, "\zeta"] & TX \arrow[d, "T\xi"] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & TX'
\end{tikzcd}$$

be a morphism of distinguished triangles. If two of the vertical morphisms, ξ, η or ζ, are isomorphisms, so is the third one.

In this thesis, the triangulated categories that we consider will be the homotopy category and the derived category of a finite dimensional algebra. More precisely, the homotopy category of an additive category $\mathcal{A}$, equipped with the shift functor and distinguished triangles isomorphic to the sequences

$$C \xrightarrow{f} D \to \mathrm{Cone}(f) \to C[1]$$

is triangulated. For any abelian category $\mathcal{A}$, the categories $D(\mathcal{A})$ as well as its bounded on the left or on the right counterparts, are triangulated with the suspension functor inherited from the corresponding homotopy category and distinguished triangles being all triangles in $D(\mathcal{A})$ (respectively $D(^{-}\mathcal{A})$, $D^{+}(\mathcal{A})$, $D^{b}(\mathcal{A})$) which are isomorphic to

$$C \xrightarrow{f} D \to \mathrm{Cone}(f) \to C[1]$$

where f is a morphism in the homotopy category [58]. The following fact implies that this triangulated structure is compatible with the abelian structure on the category of complexes.

Fact 2.2.6. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of $\mathcal{A}$ complexes. In the derived category we have an isomorphism of triangles given by the diagram below.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "p\circ\tilde{g}^{-1}"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "i"] & \mathrm{Cone}(f) \arrow[r, "p"] \arrow[u, "\tilde{g}"] & A[1]
\end{tikzcd}$$

Consider the so called *stupid truncation above i* of a complex C, defined by

$$(\sigma_{\geq i}C)_n = \begin{cases} C_n & \text{if } n \geq i \\ 0 & \text{otherwise} \end{cases} \text{ with boundaries } \partial'_n = \begin{cases} \partial_n & \text{if } n > i \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

We define the *stupid truncation below i*, $\sigma_{\leq i}C$ in a similar way.

Example 2.2.7. A complex C and its truncations above and below a certain index fit in a short exact sequence as follows

$$\begin{tikzcd}
\sigma_{\leq i-1}C \arrow[d, "f"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & C_{i-2} \arrow[r] \arrow[d] & \dots \\
C \arrow[d] & \dots \arrow[r] & C_{i+1} \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & c_{i-2} \arrow[r] \arrow[d] & \dots \\
\sigma_{\geq i}C & \dots \arrow[r] & C_{i+1} \arrow[r] & C_i \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

The sequences in each degree are clearly exact. Using Fact 2.2.6 we get that the cone of the map f is quasi-isomorphic to the truncation $\sigma_{\geq i}C$.We thus have the following distinguished triangle in the derived category

$$\sigma_{\leq i-1}C \to C \to \sigma_{\geq i}C \to \sigma_{\leq i-1}C[1]. \tag{2.5}$$

2.2.3 Structure with functors

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough projectives and injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive covariant functor. Because $F(\partial^2) = 0 = F(\partial)^2$, F induces a functor at the level of complexes defined by setting $F(C)_n = F(C_n)$ and $\partial^{F(C)} = F(\partial^C)$. If f and g are two homotopic maps from a complex C to a complex D, then their images under F are still homotopic because F is additive. Hence F also induces a functor at the level of the homotopy category. Using projective or injective resolutions we can extend F in two ways, but only on the bounded on the right respectively, on the left, derived categories. Let $F : \mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{B})$ be an additive covariant functor. The *left derived functor* $\mathbb{L}F : D^-(\mathcal{A}) \to D(\mathcal{B})$ is the composition of functors $\mathcal{N}_{\mathcal{B}} \circ F \circ \mathcal{N}_{\mathcal{A}}^{-1}$ where $\mathcal{N}_{\mathcal{B}}$ is the inclusion functor from the homotopy category to the derived category of $\mathcal{B}$ and $\mathcal{N}_{\mathcal{A}}^{-1}$ is the projective resolution functor from the derived category of $\mathcal{A}$ bounded on the left to its homotopy category. The *right derived functor* $\mathbb{R}F : D^+(\mathcal{A}) \to D(\mathcal{B})$ is constructed in a dual way. When F is induced by an additive functor at the level of complexes, its derived functors are triangulated because it sends cones of maps to cones and thus distinguished triangles to distinguished triangles.

Example 2.2.8 (derived tensor product)**.** Let (M, ∂^M) be a complex of left A-modules and (N, ∂^N) of right A-modules where A is a $\Bbbk$-algebra. Define the total complex $Tot(N \otimes M)$ by setting

$$Tot(N \otimes_A M)_k = \bigoplus_{i \in \mathbb{Z}} N_i \otimes M_{k-i}$$

with differential:

$$\partial_k^{Tot(N \otimes_A M)} = \sum_i \partial_i^N \otimes id_{M_{k-i}} + (-1)^i id_{N_i} \otimes \partial_{k-i}^M.$$

Then the left derived functor of $Tot(N \otimes_A ?) : \mathrm{Ho}^-(A\text{-mod}) \to \mathrm{Ho}^-(\Bbbk\text{-mod})$ is the *left derived tensor product*, denoted

$$N \otimes_A^{\mathbb{L}} ? : D^-(A\text{-mod}) \to D(\Bbbk\text{-mod}).$$

Example 2.2.9 (Derived Hom functor)**.** Let $\mathcal{A}$ be an abelian category and let and (M, ∂^M) and (N, ∂^N) be $\mathcal{A}$-complexes. Define the total hom complex $\mathrm{Hom}^\bullet(N, M)$ by setting

$$Hom^k(N, M) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(N_i, M_{k-i})$$

with differential $\partial^k_{\mathrm{Hom}^\bullet(N,M)}(f) = \partial^M \circ f - (-1)^k f \circ \partial^N$. Then the right derived functor of $\mathrm{Hom}^\bullet(?, M) : \mathrm{Ho}^+(\mathcal{A}) \to \mathrm{Ho}^+(\Bbbk\text{-Mod})$ is the *right derived hom functor*, denoted $\mathbb{R}\,\mathrm{Hom}_A(?, M) : D^+(\mathcal{A}) \to D(\Bbbk\text{-Mod})$.

The category of complexes over an abelian category together with the total hom functor form a dg-category [39]. Dg-categories constitute one of several more modern settings for representation theory than the one we have chosen in this thesis. However we do not mention dg-categories further because our result will only use the total hom functor in the context of the following formula which is straightforward from the definition of the total hom complex and the homotopy category.

$$H^n(\mathrm{Hom}^\bullet(M, N)) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{A})}(M, N[n]). \tag{2.6}$$

As mentioned in the introduction, this will be used in the special case where the target N is a complex concentrated in one degree and the source is a perfect complex.

Morita's theorem gave us a criterion for two algebras to have the same categories of representations. A similar theorem holds in the derived setting, though it is significantly harder to prove. The progenerating module must be replaced by the more general and more intricate *tilting object*. Fix an algebra A. An object T in $D^b(A\text{-mod})(=: D^b(A))$ is a tilting object if it is isomorphic to a complex in $\mathrm{Ho}^b(A\text{-proj})$ and

- T has no self extensions: $\mathrm{Hom}_{D^-(A)}(T, T[i]) = 0$ for all $i \in \mathbb{Z}, i \neq 0$;
- T generates the perfect derived category: the smallest triangulated full subcategory of $D^-(A\text{-Mod})$ containing all direct factors of finite direct sums of T is $\mathrm{Ho}^b(A\text{-proj})$ [61, Definition 6.1.3].

Theorem 2.2.10 (Rickard [50], Keller [41, Chapter 8])**.** *Let A and B be $\Bbbk$-algebras which are flat as modules over $\Bbbk$. The following are equivalent*

- *There is a $\Bbbk$-linear triangulated equivalence $(F, \phi) : D(A\text{-Mod}) \to D(B\text{-Mod})$;*
- *There is complex of B-A modules such that the total left derived functor*

$$X \otimes^{\mathbb{L}}_A ? : D(A\text{-Mod}) \to D(B\text{-Mod})$$

  is an equivalence of categories;

- *There is a tilting complex of B modules T such that $(\mathrm{Hom}_{D^b(A)}(T, T))^{op} \cong A$.*

Moreover,

$$\begin{aligned} D^-(A\text{-Mod}) \cong D^-(B\text{-Mod}) \Leftrightarrow & \, D^b(A\text{-Mod}) \cong D^b(B\text{-Mod}) \\ \Leftrightarrow & \, \mathrm{Ho}^b(A\text{-proj}) \cong \mathrm{Ho}^b(B\text{-proj}) \\ \Leftrightarrow & \, \mathrm{Ho}^b(A\text{-Proj}) \cong \mathrm{Ho}^b(B\text{-Proj}). \end{aligned}$$

Finally, when A and B are noetherian, the equivalence restricts to the perfect derived category

$$D^-(A\text{-Mod}) \cong D^-(B\text{-Mod}) \Leftrightarrow D^-(A\text{-mod}) \cong D^-(B\text{-mod}).$$

Note that this theorem does not give an explicit construction of X from T in the category of complexes. No such link is known in general however, a theorem of Keller holds when A is projective as a $\Bbbk$-module [61, Theorem 6.4.1].

Serre functor Poincaré duality is a classical result relating the i^{th} homology group of a closed oriented n-manifold to its $(n-i)^{th}$ cohomology group using cup product. Serre duality is a similar result for vector bundles on smooth projective varieties. Grothendieck duality generalises Serre duality to a larger class of varieties and reformulates the result through the existence of a so called Serre functor, an endofunctor of the derived category of coherent sheaves on a scheme. In this thesis, we are interested in the Serre functor in purely algebraic settings insofar as it encompasses crucial structure of the bounded derived category of modules over finite dimensional algebras. The notion was first introduced by Bondal and Kapranov [8].

A *Serre functor* $\mathbb{S}$ on a $\Bbbk$-linear triangulated category with finite dimensional hom spaces $\mathcal{T}$ is a triangulated self equivalence of $\mathcal{T}$ such that there is a bifunctorial isomorphism

$$\mathrm{Hom}(X, \mathbb{S}(Y)) \simeq D\,\mathrm{Hom}(Y, X), \forall X, Y \in \mathcal{T}.$$

From this definition, one can deduce that a Serre functor is exact and that if it exists it is unique up to natural isomorphism. The existence of a Serre functor can thus be seen as a structure on the triangulated category. In fact, it is linked to other well-known structures as a Serre functor exists *if and only if* all *homological functors* are representable.

Theorem 2.2.11 (Happel [29])**.** *Let A be a finite dimensional algebra with finite global dimension. Then the left derived functor of the Nakayama functor is a Serre functor for the bounded derived category $D^b(A\text{-mod})$.*

Later results show that the existence of a Serre functor is equivalent to the algebra having finite global dimension ([30] and [49]).

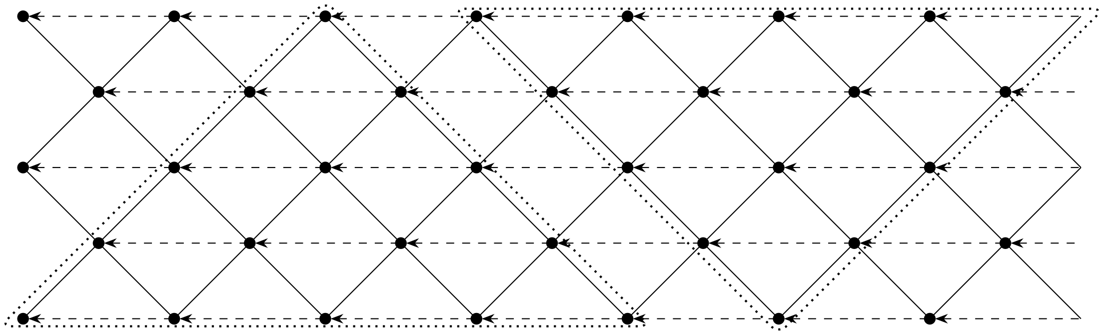


Figure 2.8: The bounded derived category of the quiver A_5

Notes 2.2.12. The derived Nakayama functor restricted to the category of modules does not act in the the same way as the original Nakayama functor. Consider once more the commutative square with a leg from Figure 2.1 where 5 is the maximum of the poset. In Example 2.1.22, we argued that $\nu S_5 = 0$. To compute $\mathbb{L}\nu(S_5)$, consider its projective resolution $0 \to P_4 \to P_5 \to 0$. Applying the original Nakayama functor in each degree we get the complex of injective modules $0 \to I_4 \to I_5 \to 0$. One can check that it has homology concentrated in degree 1 making it quasi isomorphic to $S_4[1] \ncong 0$.

We mentioned briefly at the end of Subsection 2.1.2 that the Auslander–Reiten translation is used to describe the category of finitely generated modules over a finitely generated algebra using almost split sequences. Similar results hold in the bounded derived category provided we replace almost split sequences by so called *Auslander–Reiten triangles*. In that case the derived Nakayama functor composed with a negative shift is to the derived category what the Auslander–Reiten translation is to the category of modules. It is called the Auslander–Reiten translation for that reason. It follows that the quiver of the derived category is also a translation quiver. See [29, Chapter I Section 4].

Example 2.2.13. For finite dimensional hereditary algebras, one can show that indecomposable objects of the derived category are precisely stalk complexes of indecomposable modules. Hence, expanding on example 2.1.21, Figure 2.8 [3]represents a portion of the Auslander–Reiten quiver of the bounded derived category of A_5 with the Auslander–Reiten translation indicated by dashed arrows. Note how in each copy of the category of finitely generated module the translation matches τ.

Example 2.2.14. Let us point out that the computation made in Note 2.2.12 illustrates how the image under the Serre functor is computed in general. In this thesis, this process will be used in Section 4.2 when recalling results of Yıldırım on the orbits under the Serre

[3]this drawing is quite famous see [29] for instance

functor of a family of intervals of the lattice of order ideals of the grid. The projective resolutions will be exclusively antichain resolutions.

Calabi–Yau Categories While the existence of Serre functor is in itself structural, it happens that the Serre functor behaves exceptionally well. Recall that a triangulated category $\mathcal{T}$ with Serre functor $\mathbb{S}$ is fractionally Calabi–Yau if there exist integers d and l and an isomorphism of triangulated functors

$$\mathbb{S}^l \cong [d].$$

We say that $\mathcal{T}$ is $\frac{d}{l}$-Calabi–Yau. The original setting of this property is the following example which we state without the required geometric notions for its cultural significance rather than for its usefulness in the rest of the thesis.

Example 2.2.15. Let X be a smooth projective variety of dimension n, let $D(X)$ be its derived category of coherent sheaves, K_X its *canonical bundle*. Then the functor $K_X \otimes ?[n]$ is a Serre functor on $D(X)$ and X is n-Calabi–Yau if and only if the canonical bundle is trivial.

As Kontsevich says in his notes, finite dimensional algebras can be fractionally Calabi–Yau but not Calabi–Yau without contradicting the existence of the Serre functor. See the results cited after Theorem 2.2.11. Progress has been made toward classifying fractionally Calabi–Yau categories, as for instance the following theorem on abelian hereditary categories.

Theorem 2.2.16 ([52])**.** *Let A be an indecomposable abelian hereditary category which is fractionally Calabi–Yau, then A is derived equivalent to either*

- *the category of finite dimensional representations* $\operatorname{rep} Q$ *over a Dynkin quiver Q, or*
- *the category of finite dimensional nilpotent representations* $\operatorname{nilp} \tilde{A}_n$ *where $\tilde{A}_n$ $(n > 0)$ has cyclic orientation, or*
- *the category of coherent sheaves* $\operatorname{coh} X$ *where X is either an elliptic curve or a weighted projective line of tubular type.*

Note that some of the categories mentioned in this theorem are not *a priori* linked to geometry. As an example, on Figure 2.8 we can see how the Auslander–Reiten translation applied six times yields two times the negative shift functor for equioriented type A_5

quiver algebra. This makes the algebra $\frac{4}{6}$-Calabi–Yau as the Serre functor is the shifted Auslander–Reiten translation like we mentioned above.

2.3 Higher Representation Theory

In Subsection 2.1.2 we recalled the language of classical representation theory which used short exact sequences. In the case of hereditary algebras of finite representation type, we mentioned the surprising connection with Auslander algebras which have low homological dimension. In the past 20 years, there has been efforts made towards elaborating higher homological dimensional versions of these concepts. In this section we give a short account of the starting point of higher representation theory with the goal of giving some weight and context to Theorem E. In particular, our presentation will be independent of the derived setting we introduced in the previous section.

As a motivating result, we want to describe a higher representation theoretic version of Theorem 2.1.20. To do so we need higher homological versions of the additive generator of the module category M and of the Auslander algebra. For the Auslander algebra it is quite straightforward: a finite dimensional algebra Γ is *n-Auslander algebra* if it satisfies the following inequalities

$$\operatorname{gldim}\Gamma \leq n+1 \leq \operatorname{domDim}\Gamma.$$

The generalisation of the additive generator is more subtle. *Cluster tilting objects* appeared in categorifications of cluster algebras. First we define *cluster tilting categories*. Let $n \geq 1$ be an integer. Let $\mathcal{X}$ be an extension closed subcategory of Λ-mod and let $\mathcal{C}$ be a subcategory of $\mathcal{X}$. We call $\mathcal{C}$ n-cluster tilting if it is functorially finite and

$$\begin{aligned} \mathcal{C} &= \{X \in \mathcal{X} \,|\, \operatorname{Ext}^i_\Lambda(X, \mathcal{C}) = 0 \text{ for } 0 < i < n\} \\ &= \{X \in \mathcal{X} \,|\, \operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0 \text{ for } 0 < i < n\}. \end{aligned}$$

A *cluster-tilting module* M is a module such that $\operatorname{add} M$ is an n-cluster tilting cateogry. An *absolute cluster tilting category* (resp. absolute cluster tilting module) is a cluster tilting category (resp. cluster tilting module) relative to Λ-mod. Iyama first studied cluster tilting objects to construct n-analogues of Auslander–Reiten theory where short exact sequences are replaced by n-exact sequences [34]. In [31] the authors show that if a cluster tilting object exists and the global dimension of the given algebra is bounded by n, then the cluster tilting object is unique. Call *weak n-representation finite* algebras which admit an absolute n-cluster tilting objects and *n-representation finite* a weak n-representation finite algebra with global dimension bounded by n. Note that the bound on the global dimension gives the n-analogue of being hereditary. The following theorem shows in what way the two notions we presented are the correct generalisations.

Theorem 2.3.1 (($n+1$)-Auslander correspondance [33]). *For any $n \geq 1$, there exists a bijection between the set of Morita equivalence classes of n-representation finite algebras, and the set of Morita-equivalence classes of $(n+1)$-Auslander algebras. It is given by $\Lambda \mapsto \Gamma := \mathrm{End}_\Lambda(M)$ for an n-cluster tilting object M of Λ.*

It seems natural to ask whether higher Auslander algebras can in turn be higher representation finite. This turns out to be quite rare. First we discuss results about a more general class of algebras. In [36], the author considers *n-complete* algebras. We do not give the precise definition but point out that n-complete algebras have n-cluster tilting categories and that n-representation finite algebras are n-complete.

Theorem 2.3.2 (Iyama [36, Theorem 1.14]). *Let Λ be an n-complete algebra. Let M be an n-cluster tilting. Then $\mathrm{End}_\Lambda M$ is $(n+1)$-complete.*

In particular, if Λ is n-representation finite then its associated $(n+1)$-Auslander algebras has an $(n+1)$-cluster tilting category. It is not absolute in general [36, Theorem 1.19]. By [36, Theorem 1.14], for a 1-representation finite algebra Λ we can construct inductively a tower of higher complete algebras denoted $\Lambda^{(d)}$ [36, Corollary 1.16].

Example 2.3.3. Let n and d be integers. Take as the base of our tower the representation finite algebra of equioriented type A_n. The tower of n-complete algebras $A_n^{(d)}$ constructed using [36, Theorem 1.14] is in fact a tower of n-representation finite algebras by [36, Theorem 1.19]. They are called the Higher Auslander algebras of type $A_n^{(d)}$. See Chapter 5.3 for the explicit description. This is a special case of a neat construction for n-complete algebras which works for all finite dimensional algebras of type ADE [36, Theorem 6.12].

2.4 A conjecture of Chapoton

In this section we give more details regarding a conjecture of Chapoton linking combinatorial formulas, fractionally Calabi–Yau posets and symplectic geometry which was mentioned briefly in the introduction.

2.4.1 Geometric notions

We recall certain geometric notions from [46]. A function $f : \mathbb{C}^n \to \mathbb{C}$ is *quasi homogeneous of weights $d_1, \ldots, d_n$ and total degree D* if for all $(x_i)_{i\leq n} \in \mathbb{C}$ and $\lambda \in \mathbb{C}^n$, we have

$$f(\lambda^{d_1} \cdot x_1, \ldots \lambda^{d_n} \cdot x_n) = \lambda^D \cdot f(x_1, \ldots, x_n). \tag{2.7}$$

We consider the zero locus of f *i.e.* its *algebraic variety*. We say that $x \in \mathbb{C}^n$ is a *singular point* of f if all the partial derivatives of f vanish at x. A singular point is *isolated* if there exists a neighbourhood of x in which no other point is singular. From now on we consider a quasi homogeneous polynomial f with a unique singular point at the origin and call its corresponding variety an *isolated singularity*. Its *Milnor number*, denoted μ_f is the dimension of its n^{th} cohomology group. Its *characteristic polynomial* is the characteristic polynomial of the *monodromy map* which we do not define here. In [46] the authors compute explicitly the characteristic polynomial and the Milnor number of a quasi homogeneous isolated singularity f. We give here their formula for the latter and refer the reader to the original source for more details.

$$\mu_f = \prod_{i=1}^{n} \frac{D - d_i}{d_i}. \tag{2.8}$$

Fukaya categories are categories whose objects are *Lagrangian submanifolds* of a singularity and whose morphism spaces are characterized by combinatorics on the *intersections* of these submanifolds. To make this precise requires a lot of work [53] and we will stay away from it.

2.4.2 From weights to Fukaya categories

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative integers. We fix n and assume that there exists non negative integers $m^n, d_1^n, \ldots, d_{m^n}^n$ and D^n such that the following equation hold

$$s_n = \prod_{i=1}^{m^n} \frac{D^n - d_i^n}{d_i^n}. \tag{2.9}$$

Please note the similarity with equation (2.8). To simplify notation we will omit the n. We call *weights* of s_n the data $(d_1, \ldots, d_m; D)$. We call the above equation a *product formula*. Every sequence of integers has at least one product formula.

Example 2.4.1. Any sequence of integers $(s_n)_{n\in\mathbb{N}}$ can be written

$$s_n = \frac{s_n}{1} \tag{2.10}$$

which gives the weight $(1; s_n + 1)$ for all n. We call this the *trivial* product formula.

The following examples capture the intention behind the definition of product formulas and weights. In particular a given sequence can have more than one product formula.

Example 2.4.2. Consider the Catalan sequence defined for all $n \in \mathbb{N}$ by $c_n = \frac{1}{n+1}\binom{2n}{n}$. Catalan numbers have a product formula as follows

$$c_n = \prod_{i=2}^{n+1} \frac{2n+2-i}{i} \tag{2.11}$$

with weights $(2, \dots, n+1; 2n+2)$ or $(2, \dots, n; 2n+2)$ as the fraction $\frac{n+1}{n+1}$ can either be kept or omitted. We will come back to discuss this choice after having discussed Chapoton's conjecture in its entirety.

Example 2.4.3. Fix a positive integer m. As mentioned in the introduction the main example of interest for this thesis is the sequence $\binom{m+n}{m}_{n\in\mathbb{N}}$ which has weights $(1, \dots, m; m+n+1)$ or $(1, \dots, n; m+n+1)$. Note that the roles of m and n are symmetric.

Conjecture 2.4.4. *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative integers. Suppose that for all n there exists weights $(d_1, \dots, d_m; D)$ for s_n. Then there should exist a family of finite posets $(P_n)_{n\in\mathbb{N}}$ such that for all n, P_n has cardinal s_n and the bounded derived category of P_n is $\frac{C}{D}$-Calabi–Yau where the integer C is given by the formula*

$$C = \sum_{i=1}^{m} D - 2d_i.$$

Moreover, $D^b(P_n)$ should be equivalent to a type of Fukaya category associated with a generic isolated quasi-homogeneous singularity with variable weights $(d_1, \dots d_m)$ and total weight D. In particular, the Milnor number and the characteristic polynomial of f are determined by the weights of the sequence using Milnor and Orlik's formula. Furthermore this correspondance should be compatible with the natural monoidal structures at play, namely with cartesian product of posets, tensor product of triangulated categories, and the monoid structure on Weights.

Here is a short historical timeline of the progress made towards this conjecture.

1998 Maxime Kontsevich defines the notion of fractionally Calabi–Yau for triangulated categories with a Serre functor categories in a course at the École Normale Supérieure in Paris [42].

1999 Jun-Ichi Miyachi and Amnon Yekutieli show that the bounded derived categories of Dynkin quivers are fractionally Calabi–Yau by computing their Picard groups.

2007 Chapoton shows that the Coxeter transformation of the Tamari lattice has finite order using operads [12].

2012 Chapoton conjectures, among other things that the bounded derived category of the Tamari is fractionally Calabi–Yau [13].

2018 Yıldırım show that the Coxeter transformation of the order ideals of cominuscule posets of type A, B, D and E are of finite order [60].

2021 Rognerud shows that the bounded derived category of the Tamari lattice is fractionally Calabi–Yau using exceptional intervals [51].

2023 Chapoton publishes his conjecture on fractionally Calabi–Yau posets [14].

2024 This author proves that the bounded derived category of the order ideals of grids are fractionally Calabi–Yau and derived equivalent to some Fukaya Seidel Categories related to type A [27].

More examples satisfying these conditions, for which the conjecture has no answer so far, can be found in [14]: alternating sign matrices, green mutation posets for cyclic quivers, the West Family, the Tamari-intervals family as well as a plethora of small examples.

Notes 2.4.5. Like with any unproven conjecture, one can wonder whether the current formulation of Chapoton's encompasses the correct scope. Here are some questions that I find relevant but have not had the opportunity to explore further. Some of them were asked during talks given about this topic at the GATo seminar in Amiens or at the CHARMS summer school in Versailles in the summer of 2024.

- Given the examples that have been proven so far, can we expect the posets P_n to be lattices? Or even semidistributive lattices?

- Because there can be more than one product formula on a sequence, are there product formulas that are better than others? Is it possible to rank the product formulas of a given sequence in a systematic way? Can we say somehow that one of the weights for the Catalan sequence from Example 2.4.2 is better than the others including the trivial weight?

- Is it possible that we should only consider sequences where the integer m depends closely on n, say $m = n$?

Finally, we would like to record here a remark made by Chapoton at the CHARMS conference in Strasbourg in 2024: when $D = 2k$, adding a factor $\frac{k}{k}$ is not very important from a geometric point of view.

Chapter 3

Antichain modules

This section contains technical results about certain classes of antichain modules, their morphisms and extensions. The main one gives a way to study the fractionally Calabi-Yau property on lattices. Once the correct antichains are identified, the proof is formal. Some lemmas rely on observations about (co)simplicial sets [54, Tag 019H] and their associated chain complexes.

3.1 Boolean antichains

Let C be an antichain of size r in a lattice L and M_C^α its associated antichain module below $\alpha \in L$. Note that in degree i of the projective resolution $\mathcal{P}_C^\alpha$ of M_C^α there are $\binom{r}{i}$ indecomposable components in direct sum. If one forgets the modules, the complex has the shape of the power set of C, however the indices of the modules in each degree are not necessarily in bijection with the lattice $(\mathcal{P}(C), \subseteq, \cup, \cap)$ (see Figures 3.1 to 3.3).

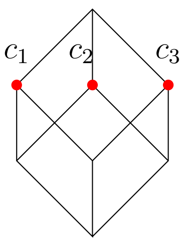


Figure 3.1: Boolean antichain

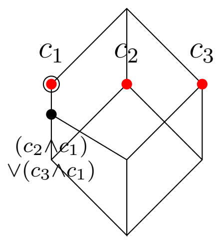


Figure 3.2: Strong, not intersective, antichain

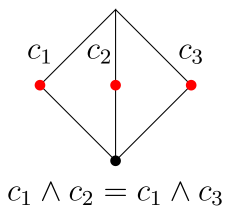


Figure 3.3: Antichain which is neither

Examples and non examples for key properties of antichains

To make this statement precise, let us introduce four conditions on C as an antichain of $[\hat{0}, \alpha]$ for some $\alpha \in L$.

Inclusive antichain For all subsets S and S' of C, if $\wedge S \leq \wedge S'$ then $S' \subseteq S$.

Intersective antichain For all subsets S and S' of C, we have $(\wedge S) \vee (\wedge S') = \wedge(S \cap S')$.

Strong antichain For all S, S' subsets of C of same cardinal, $\wedge S$ and $\wedge S'$ are incomparable *i.e.* if $\wedge S \leq \wedge S'$ then $S = S'$.

Boolean antichain C is both inclusive and intersective.

See Figures 3.2 and 3.1 for examples of strong, intersective and boolean antichains in different small lattices. Figure 3.3 gives a non example. Figures 3.2 shows how a strong antichain can be not intersective. Please note that intersectivity depends on the choice of a top element α whereas inclusivity and strength do not. This is essentially because the join of the empty set is the maximum of the ambient lattice. Hence it is important to compute the meet and join operations in the interval $[\hat{0}, \alpha]$. Note also the following lemma.

Lemma 3.1.1. *An antichain is* **inclusive** *if and only if it is* **strong**.

Proof. The inclusivity condition implies the strong antichain condition by taking the subsets S and S' with the same cardinal. To see the converse, assume that the antichain C is a strong antichain and let S and S' be two subsets of C such that $\wedge S' \leq \wedge S$. Suppose at first that $|S| + n = |S'|$ with $n > 0$. Then there exist $s_1, \ldots, s_n \in S' \setminus S$. Set $S'' = S \sqcup \{s_1, \ldots, s_n\}$. Because the inequalities $\wedge S' \leq \wedge S$ and $\wedge S' \leq \wedge\{s_1, \ldots, s_n\}$ hold, we have

$$\wedge S' \leq (\wedge S) \wedge (\wedge\{s_1, \ldots, s_n\}) = \wedge S''.$$

Because $|S'| = |S''|$ and the antichain is strong, we have $S' = S''$ and so $S \subseteq S'$. Next if $|S| = |S'| + n$ with n positive, then take $s_1, \ldots, s_n$ in $S \setminus S'$. Then we have

$$\wedge S'' := (\wedge S') \wedge (\wedge\{s_1, \ldots, s_n\}) \leq \wedge S.$$

The antichain is strong so $S = S''$. Then $S' \subseteq S$ so $\wedge S' \geq \wedge S$ and thus $\wedge S' = \wedge S$. Using the first part of the proof we get $S = S'$. This contradicts the assumption on the integer n. $\square$

Remark 3.1.2. The strong antichain condition implies that for each n, the set

$$\{\wedge S | S \subseteq C \text{ with } |S| = n\}$$

is an antichain. This condition is strong enough for the main result (see 3.5). However it is not strong enough for computing morphisms sets.

Denote $\langle C\rangle^{\alpha}_{\vee,\wedge}$ the lattice generated by the elements of C and α in the sublattice $[\hat{0}, \alpha]$ of L, equipped with the lattice operations of L. The following lemma motivates the terminology.

Lemma 3.1.3. *An antichain is boolean if and only if the map*

$$\begin{array}{ccc} (\mathcal{P}(C), \cap, \cup) & \xrightarrow{\phi} & (\langle C\rangle^{\alpha}_{\vee,\wedge}, \wedge, \vee) \\ S & \mapsto & \wedge S \end{array}$$

is a lattice anti-isomorphism.

Proof. Assume that the map ϕ is a lattice anti-isomorphism. Then C_α is intersective because ϕ sends $\cap$ to $\vee$. Now consider $S, S' \subseteq C$ such that $\wedge S \leq \wedge S'$. This is equivalent to the following equality

$$\wedge S = (\wedge S) \wedge (\wedge S').$$

The right hand side is equal to $\wedge(S \cup S')$. Because ϕ is a bijection, $S = S \cup S'$ meaning that $S' \subseteq S$. Thus C is inclusive. Conversely assume C is both inclusive and intersective below α. The fact that ϕ sends $\cup$ to $\wedge$ is true for any subset of a lattice. The intersection property makes ϕ send $\cap$ to $\vee$. To see that ϕ is injective, note that if $\wedge S = \wedge S'$ then the inclusion property forces $S = S'$. To see that the map is surjective, notice that the image of ϕ, $\mathrm{Im}(\phi) = \{\wedge S | S \subseteq C\}$ is a lattice, using the properties we just exhibited. Moreover, any sublattice of $[\hat{0}, \alpha]$ containing C contains $\mathrm{Im}(\phi)$. It is thus the sublattice of $[\hat{0}, \alpha]$ generated by C and α, *i.e.*, $\langle C\rangle^{\alpha}_{\vee,\wedge} = \{\wedge S | S \subseteq C\}$ and ϕ is surjective. □

Remark 3.1.4. Following a question from one of the reviewers of this thesis, it became apparent that if an antichain is intersective below an element $\alpha \in L$ then it should be automatically strong as long as it is not the singleton $\{\alpha\}$. In particular, intersective and boolean should be the same thing in all of the cases we consider. This observation would make the first part of the previous proof slightly shorter as only the intersectivity needs to be checked. This would not change the statements or the proofs that follow. Moreover, the distinction between strong and boolean antichains remains valid. In order to keep the changes to this manuscript minor, we do not include the proof of this new claim here. However, we can produce it upon request and intend to include it in further works when needed.

3.2 Global dimension

In this section we use the above vocabulary in the context of join semidistributive lattices. The importance of join semidistributive lattices over meet semidistributive lattices comes from our choice of orientation for the Hasse diagram. If we had picked arrows to go the other way around, we would have had a result on meet semidistributive lattices. With the current conventions, what holds is a dual result for injective resolutions.

Theorem 3.2.1. *Let L be a finite join semidistributive lattice and $\Bbbk$ a field. Then the simple modules over $\mathcal{A}_{\Bbbk}(L)$ are strong antichain modules.*

As a corollary we extend a result from [37].

Corollary 3.2.2. *Let L be a finite join semidistributive lattice with at least two elements. The global dimension of the incidence algebra of L is given by the maximal covering number of L.*

Join semidistributive lattices appear in representation theory as lattices of cluster tilting objects or of torsion pairs [17]. The proof of the result is a straightforward generalisation of the proof in [37], but it does apply to classes of interesting new examples. Moreover just like in [37], the reason why this result is interesting is mostly the following corollary. Unlike in [37], because the lattice is not distributive, the maximal covering number is not the order dimension of the lattice.

Corollary 3.2.3. *The global dimension of the incidence algebra of a semidistributive lattice over a field $\Bbbk$ is independent of the field.*

For the convenience of the following proof we say that a subset S of an antichain C is *minimal* if for any subset S' of S, we have $\wedge S' > \wedge S$. As the lattice is finite, such a minimal set, while not unique, but can always be obtained by removing elements from S.

Proof of Theorem 3.2.1. Let L be a lattice and let x be an element of L. Consider the set $C = \{c_1, \dots, c_n\}$ of elements of L covered by x. They form an antichain and its associated antichain module below x is the simple module S_x. Let S_1 and S_2 be non empty subsets of C and assume that $\wedge S_1 \leq \wedge S_2$ while $S_2 \not\subseteq S_1$ *i.e.* that the antichain is not strong. We will show that the lattice is not join semidistributive. We assume S_1 is minimal. If S_1 has only one element, then $\wedge S_1$ is covered by x so $\wedge S_1 = \wedge S_2$ and $S_2 = S_1$ because C is an antichain. So we assume that S_1 has at least two elements. By our original assumption, there exists $k \in S_2 \setminus S_1$. Hence for $c \in S_1$, we have $c > \wedge S_1$ and $\wedge(S_1 \setminus \{c\}) > \wedge S_1$. If moreover $\wedge(S_1 \setminus \{c\}) \not\leq k$, then we are done:

$$\wedge(S_1 \setminus \{c\}) \vee k = x = c \vee k > k \vee (\wedge(S_1 \setminus \{c\}) \wedge c) = \wedge S_1 \vee k = k.$$

However, it is not always true that $\wedge(S_1 \setminus \{c\}) \not\leq k$. If not, replace S_1 by $S_1 \setminus \{c\}$ and apply the same reasoning. Because $S_1 \setminus \{c\}$ is strictly smaller than S_1 and the antichain is finite, this process ends either when $\wedge(S_1 \setminus \{c\}) \not\leq k$ or when $S_1 = \{c_1, c_2\}$. In that case, the elements of C are covered by x, S_1 is minimal and we have $k \vee c_1 = k \vee c_2 > k \vee (c_1 \wedge c_2)$. Like before this means that the lattice is not join semidistributive. $\square$

The next proposition is true for any lattice and is one of the interesting upshots of the definition of strong antichains.

Proposition 3.2.4. *Antichain resolutions of strong antichains are minimal.*

Proof. Let C be a strong antichain. To see that the antichain resolution $\mathcal{P}_C$ is minimal, we need to argue that the maps

$$\bigoplus_{|S|=k} P_{\wedge S} \xrightarrow{\partial_k} \mathrm{Im}(\partial_k)$$

are projective covers for each degree k. In other words we show that the kernels of these maps are superfluous. Fix k. If the kernel is zero, there is nothing to prove. Hence assume that it is not and consider a module N such that

$$\mathrm{Ker}(\partial_k) + N = \bigoplus_{|S|=k} P_{\wedge S}$$

The module on the right is generated by the elements $e_{\wedge S}$ with $|S| = k$. Because the antichain is strong, the elements $\wedge S$ with $|S| = k$ cannot be compared and they each appear in a different direct summand. Moreover, because the quiver has no oriented cycles, the only path that goes from $\wedge S$ to $\wedge S$ is the lazy path. Hence for each S with $|S| = k$, $e_{\wedge S}$ is in $\mathrm{Ker}(\partial_k)$ or in N as it cannot be written as the sum of two elements.

Because we assume that the kernel is non empty and we are considering a projective resolution, we know that $\mathrm{Ker}(\partial_k) = \mathrm{Im}(\partial_{k+1})$. Recall from equation (1.6) that the image is generated by paths associated to relations $\wedge S \leq \wedge(S - \{i\})$ of the posets. Because the antichain is strong we always have $\wedge S < \wedge(S - \{i\})$ so that the lazy paths are not in $\mathrm{Ker}(\partial_k)$. We conclude that they are in N and that $N = \bigoplus_{|S|=k} P_{\wedge S}$, completing the proof. $\square$

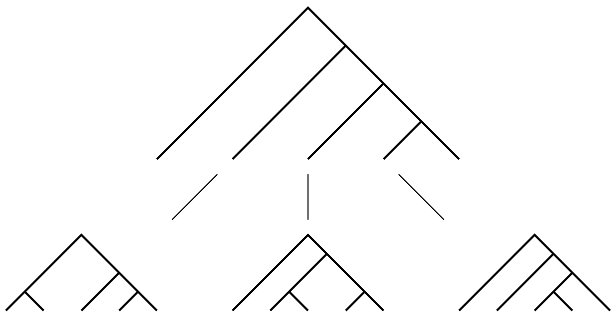


Figure 3.4: Covering relations from the maximum of Tamari 4

Proof of Corollary 3.2.2. The arguments for this proof are identical to those of [37]. We reproduce them here for the convenience of the reader. Let x be in L. The simple module S_x has an antichain resolution of length $\mathrm{cov}(x)$, the size of its associated antichain. By Theorem 3.2.1 this antichain is strong and by Proposition 3.2.4 the resolution is minimal. The projective dimension of the simple module S_x is the covering number of x. Hence, as recalled in Theorem 2.1.16 $\mathrm{gldim}(L) = \max_{x \in L}(\mathrm{pdim}\, S_x) = \mathrm{cov}(L)$. □

Example 3.2.5. Recall from Example 2.1.37 the Tamari lattice which is semidistributive so in particular join semidistributive. Using its description in terms of binary trees ordered with tree rotations, it is easy to see that from each tree there are at most $n-1$ possible tree rotations going up and down from the vertex. Moreover, there is one element of the lattice which has $n-1$ coverings, that is the minimum of Tam_n. Dually, the maximum is the unique element of the lattice which covers exactly $n-1$ trees. See Figure 3.4. It follows that $\max\{|\mathrm{cov}(x)| \text{ for } x \in \mathrm{Tam}_n\} = n-1 = \mathrm{gldim}(\mathcal{A}_{\Bbbk}(\mathrm{Tam}_n))$.

Example 3.2.6. The lattice $J_{m,n}$ is distributive as a lattice of order ideals. The result of [37] suffices to compute the global dimension. In 2.1.35 we showed that $\mathrm{cov}(J_{m,n}) = \min(m,n)$ which is also the order dimension. Hence the global dimension of $J_{m,n}$ is $\min(m,n)$.

3.3 Morphisms

In this section we fix an antichain C below an element α of a lattice L as well as an interval $I = [a,b]$ of L. If C is *boolean* more can be said about the total hom complex $\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathcal{P}^\alpha_C, I)$. Recall the notation from equation (1.4). Note that

$$e_x \cdot I = \begin{cases} \Bbbk & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

as a special case of equation (1.5). Denote by E the set $\{S \subseteq C | \wedge S \in I\}$. A projective module $P_{\wedge S}$ appearing in the projective resolution $\mathcal{P}^{\alpha}_{C}$ can contribute to the total hom if and only if $S \in E$. Note that C is finite hence its set of subsets is finite. Assume that E is not empty otherwise the total hom complex is zero. If the antichain is intersective, and S and S' are in E then

$$a \leq \wedge(S \cup S') \leq \wedge(S \cap S') \leq b$$

meaning that $S \cup S'$ and $S \cap S'$ are in E as well. It follows that E has a largest element $\cup_{S \in E} S = S_M$ as well as a least element $\cap_{S\in E} S = S_m$. Moreover, if $S \in [S_m, S_M]$ then

$$a \leq \wedge S_M \leq \wedge S \leq \wedge S_m \leq b$$

hence $S \in E$. There are no other elements in E and we have proved the following lemma.

Lemma 3.3.1. *If C is a boolean antichain then $E = [S_m, S_M]$.*

Remark 3.3.2. Following Remark 3.1.4 we point out that the fact that C is inclusive should be a concequence of the fact that C is intersective which is in turn equivalent to C being boolean. This remark does not change the validity of the statement above.

To describe the total hom complex, $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(\mathcal{P}^{\alpha}_{C}, I)$, we write $m = |S_m|$ and $M = |S_M|$. For each degree $m \leq i \leq M$ there are exactly $\binom{M-m}{i-m}$ subsets S of C with cardinal i in E. Hence, by equations (1.4) and (3.1) the complex $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(\mathcal{P}^{\alpha}_{C}, I[0])$ has shape:

$$0 \leftarrow \Bbbk^1 \leftarrow \cdots \leftarrow \Bbbk^{\binom{M-n}{j}} \leftarrow \cdots \leftarrow \Bbbk \leftarrow 0. \tag{3.2}$$

This is precisely the shape of the simplicial module associated to the standard simplex. It remains to see that the boundary maps match the standard simplex as well. In each degree the map is post composition by the boundary map of $\mathcal{P}^{\alpha}_{C}$. It sends a map defined by a vector $(f_S)_{|S|=i}$ to a vector $(g_{S'})_{|S'|=i+1}$ described by

$$g_{S'} = \begin{cases} 0 & \text{if } S \not\subseteq S', \\ (-1)^{\epsilon(S,x)} \cdot f_S & \text{otherwise,} \end{cases}$$

where $|x|_S = |\{y \in S' | y \leq x\}|$. Indexing the vector elements by their complements in C and writing the basis vectors e_J we get

$$e_J \mapsto \sum_{x \in J} (-1)^{|x|_{J^c}} e_{J - \{x\}}$$

Like in the proof of [37, Theorem 2.2], we have an isomorphism

$$\mathrm{Hom}^{\bullet}_{\mathcal{A}}(\mathcal{P}^{\alpha}_{C}, I) \cong (\Bbbk \xleftarrow{id} \Bbbk)^{\otimes(M-m)}[m]. \tag{3.3}$$

The left hand side is a Koszul complex over the base field $\Bbbk$. As a tensor product of acyclic complexes, using Künneth's formula ([10, Chapter 6.3] or [59, Exercise 1.2..5]), it is either acyclic or concentrated in one degree when $S_M = S_m$.

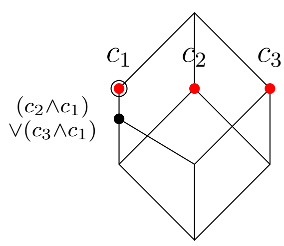
Proposition 3.3.3. *Let C be an antichain of a lattice L and let $I \subseteq L$ be an interval. Suppose the set $E = \{S \subseteq C | \wedge S \in I\}$ is an interval of the lattice $\mathcal{P}(C)$. Then there exists at most one integer p such that $\mathrm{Hom}_{\mathrm{Ho}}(M_C, I[p])$ is non zero. When such an integer exists, the hom space is one dimensional.*

Proof. Given the previous calculations and remarks, the result follows from equation (1.3). □

Moreover we know that such a degree exists if and only if the set E is a singleton *i.e.* there exists a unique $S \subseteq C$ such that $\wedge S \in I$. In this case $p = |S|$. Combining this proposition with Lemma 3.3.1 and the isomorphism u of equation (1.3) we have proved the following theorem.

Theorem 3.3.4. *Let C be a boolean antichain of a lattice L . Let $I \subseteq L$ be an interval. There exists at most one integer p such that $\mathrm{Hom}_{D^b}(M_C, I[p])$ is non zero. When such an integer exists, the hom space is one dimensional.*

Example 3.3.5. Consider the lattice in Figure 3.2, that we reproduce here for the convenience of the reader and the strong antichain $C = \{c_1, c_2, c_3\}$ below $\hat{1}$.



Its associated module is the simple $S_{\hat{1}}$ and its antichain complex is given by the following diagram. The red arrows represent the components of the boundary maps.

When the sign is not indicated, it is positive.

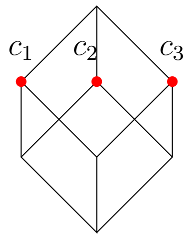
$$
\begin{tikzcd}
 & & P_{c_1\wedge c_2} \arrow[r, dashed] \arrow[ddr, dashed] & P_{c_1} \arrow[ddr, dashed] & & \\
 & & \oplus & \oplus & & \\
0 \arrow[r] & P_{c_1\wedge c_2\wedge c_3} \arrow[r] \arrow[uur, dashed, "-"] \arrow[r, dashed, bend left] \arrow[ddr, dashed, "-"] & P_{c_1\wedge c_3} \arrow[r] \arrow[uur, dashed, "-"'] \arrow[ddr, dashed, "-"'] & P_{c_2} \arrow[r] \arrow[r, dashed, bend left] & P_{\hat{1}} \arrow[r] & 0 \\
 & & \oplus & \oplus & & \\
 & & P_{c_2\wedge c_3} \arrow[r, dashed, "-"] \arrow[uur, dashed] & P_{c_3} \arrow[uur, dashed] & &
\end{tikzcd}
\tag{3.4}
$$

We consider the element $x = (c_1 \wedge c_2) \vee (c_1 \wedge c_3)$ and the interval $I = [c_1 \wedge c_2, x]$ of the lattice. Please note that this interval is in fact the set $\{c_1 \wedge c_2, x\}$ Hence the set $E = \{S \subseteq C | \wedge S \in I\}$ is the singleton $\{\{c_1, c_2\}\}$ which is an interval. So Proposition 3.3.3 applies and $\dim \mathrm{Hom}_{\mathrm{Ho}}(\mathcal{P}_C, I[2]) = 1$ while for any other shift of the interval it is 0. Any non zero morphism in degree two is proportional to the following.

$$
\begin{tikzcd}
P_{\hat{0}} \arrow[r] \arrow[d] & P_{c_1\wedge c_3} \oplus P_{c_1\wedge c_2} \oplus P_{c_2\wedge c_3} \arrow[r] \arrow[d, "P_{c_1\wedge c_2}\to I"] & P_{c_1} \oplus P_{c_2} \oplus P_{c_3} \arrow[r] \arrow[d] & P_{\hat{1}} \\
0 \arrow[r] & I \arrow[r] & 0 &
\end{tikzcd}
$$

By equation (1.5), and because no other subset S' of C is in E, this map is not zero homotopic. Moreover no morphism of degree different from two exists.

Example 3.3.6. Consider the lattice from Figure 3.1, which we reproduce here for the convenience of the reader, and the antichain $C = \{c_1, c_2\}$ below $\hat{1}$. Its associated antichain



module is the interval $[c_3, \hat{1}]$. Consider moreover the interval $I = [\hat{0}, c_1]$. Notice that the ambient lattice is isomorphic to the power set of a set with three elements. The sublattice spanned by the elements of the antichain and $\hat{1}$ is the interval $[c_1 \wedge c_2, \hat{1}]$. By Lemma 3.1.3, the lattice is boolean. Theorem 3.3.4 applies. We can check that the set $E = \{S \subseteq C | \wedge S \in I\}$ is the interval $[c_1 \wedge c_2, c_1]$. In that case, $\dim \mathrm{Hom}_{\mathrm{Ho}}(\mathcal{P}_C, I[p]) = 0$ for all integer p as E is not a singleton. Because this example is small we can notice how any morphism of complexes from $\mathcal{P}_C$ to $I[1]$ would be forced to be zero by the non zero composition of non zero morphisms of intervals $P_{c_1\wedge c_2} \to P_{c_1} \to I$. Additionally, while there exists a non zero morphism of complexes from $\mathcal{P}_C$ to $I[2]$ it is made zero homotopic by the non zero morphism of intervals $P_{c_1} \to I$.

3.4 Truncations

Recall the stupid truncation $\sigma_{\geq i} R$ of a complex $R = ((R_n)_{n\in\mathbb{Z}}, (\partial_n)_{n\in\mathbb{Z}})$ defined in equation (2.4) In this section we fix a *strong* antichain C. To make notation lighter we do not say if it is below some α, though the two lemmas below hold in both $[\hat{0}, \alpha]$ and L.

Lemma 3.4.1. *With C as above and notation from the previous sections, for all $r = |C| \geq i \geq 0$ there is a bijection*

$$\begin{aligned} \Phi : \mathrm{End}_{\mathcal{A}}((\mathcal{P}_C)_{i-1}) &\to \mathrm{Hom}_{\mathrm{Ho}}(\sigma_{\geq i}\mathcal{P}_C, (\mathcal{P}_C)_{i-1}[i]) \\ f \qquad &\mapsto \qquad f \circ \partial_i[i]. \end{aligned}$$

Proof. If $i = 0$, $\mathcal{P}_{i-1} = 0$. We assume $r \geq i > 0$. The antichain is strong so the indices of the indecomposable summands of $(\mathcal{P}_C)_{i-1} = \bigoplus_{|S|=i-1} P_{\wedge S}$ cannot be compared. Thus the endomorphisms of this module decompose as

$$f = \bigoplus_{|S|=i-1} \lambda_S \cdot id_{P_{\wedge S}}. \tag{3.5}$$

By projecting on the summands of the target, it suffices to show that there is a bijection

$$\begin{aligned} \Phi : \mathrm{End}(P) &\to \mathrm{Hom}_{\mathrm{Ho}}(\sigma_{\geq i}\mathcal{P}_C, P[i]) \\ f \qquad &\mapsto \qquad f \circ \pi_P \circ \partial_i[i] \end{aligned}$$

for all $P = P_S$ with $S \subseteq C$ of cardinal $i - 1$. Write $S = \{s_1, \ldots, s_{i-1}\}$. The morphisms on the right hand side are of the form:

$$\begin{tikzcd} \cdots & \bigoplus\limits_{|S'|=i+1} P_{\wedge S'} \arrow[r, "\partial_{i+1}"] \arrow[d] & \bigoplus\limits_{|S'|=i} P_{\wedge S'} \arrow[r] \arrow[d, "\phi"] \arrow[dl, dashed, "0"] & 0 \arrow[dl, dashed, "0"] \\ & 0 \arrow[r] & P \arrow[r] & 0 \end{tikzcd} \tag{3.6}$$

The dashed arrows represent potential homotopy maps. There cannot be a non zero homotopy so

$$\mathrm{Hom}_{\mathrm{Ho}}(\sigma_{\geq i}R, P[i]) \cong \mathrm{Hom}_C(\sigma_{\geq i}R, P[i]).$$

The relation $\partial^2 = 0$ implies that Φ is well defined. An element of $\mathrm{End}(P)$ is either 0 or an automorphism. Moreover, considering the specific form of the boundary maps of the complex $\mathcal{P}_C$ in equation (1.6), the projection of ∂_i on the factor P of its target is non zero. Hence if f is non zero then $\Phi(f)$ is non zero and Φ is injective.

It remains to see that the map is surjective. Notice that there is an isomorphism between $\mathrm{Hom}_C(\mathcal{P}_C, P[i])$ and $\mathrm{Hom}_C(\sigma_{\geq i}\mathcal{P}_C, P[i])$ as any map ϕ as in equation (3.6) also yields a map from the untruncated complex to $P[i]$ and vice versa. The map Φ is surjective if and only if every element of $\mathrm{Hom}_C(\mathcal{P}_C, P[i])$ is zero homotopic. Because the antichain is strong we have

$$E = \{S' \subset C | \wedge S' \leq \wedge S\} = \{S' \subseteq C | S \subseteq S'\}$$

where E is the set of contributing subsets of C in the total hom. It is an interval. The assumptions on i and the cardinal of C ensure that E contains at least two elements. By Proposition 3.3.3 we have

$$\mathrm{Hom}_{\mathrm{Ho}}(\mathcal{P}, P_{\wedge S}[i]) \cong 0.$$

This concludes the proof. □

We consider examples on small (non distributive) lattices.

Example 3.4.2. In the diamond lattice (Figure 3.3), the antichain $\{c_2, c_3\}$ is strong. Its associated antichain module below $c_2 \vee c_3 = \hat{1}$ (the maximum of the lattice) is the interval $[c_1, \hat{1}]$. Its antichain projective resolution is

$$R : 0 \to P_{\hat{0}} \xrightarrow{\partial_2} \begin{matrix} P_{c_2} \\ \oplus \\ P_{c_3} \end{matrix} \xrightarrow{\partial_1} P_{c_2 \vee c_3} \to 0.$$

By Lemma 3.4.1, the maps from $\sigma_{\geq 1} R$ to $P_{\hat{1}}[1]$ are all proportional to ∂_1.

However, without the incomparability condition, the result fails.

Example 3.4.3. We now give a non example. In the diamond lattice (see Figure (3.3)) consider the antichain $\{c_1, c_2, c_3\}$ under $\hat{1}$. Its associated module is the simple $S_{\hat{1}}$ and its boolean resolution R has the same shape as in equation (3.4). However this time we have

$$c_1 \wedge c_2 \wedge c_3 = c_1 \wedge c_2 = c_1 \wedge c_3 = c_2 \wedge c_3 = \hat{0}.$$

So the antichain is not strong and three copies of $P_{\hat{0}}$ appear in degree 2 and one in degree 3 of this chain complex. Consider the morphism of modules

$$\phi : \begin{matrix} P_{c_1\wedge c_2} \\ \oplus \\ P_{c_1\wedge c_3} \\ \oplus \\ P_{c_2\wedge c_3} \end{matrix} \xrightarrow{\begin{bmatrix} \iota_{\hat{0}}^{c_1} & 2\cdot\iota_{\hat{0}}^{c_1} & \iota_{\hat{0}}^{c_1} \end{bmatrix}} P_{c_1}$$

One checks the equality

$$\phi\circ\partial_3 = \begin{bmatrix} \iota_{\hat{0}}^{c_1} & 2\cdot\iota_{\hat{0}}^{c_1} & \iota_{\hat{0}}^{c_1} \end{bmatrix} \times \begin{bmatrix} -id_{\hat{0}} \\ id_{\hat{0}} \\ -id_{\hat{0}} \end{bmatrix} = 0$$

meaning that ϕ defines a morphism of complexes concentrated in degree 2 from $\sigma_{\geq 2}R$ to P_{c_1} which is a summand of R_1. The map ϕ has support all three copies of $P_{\hat{0}}$ present in degree 2 whereas $\pi_1\circ\partial_2$ is supported only by the copies associated to $c_1\wedge c_2$ and $c_1\wedge c_3$. This means that ϕ cannot be factored through $\pi_1\circ\partial_2$.

Lemma 3.4.4. *Let C be a strong antichain. Then*

$$\dim \mathrm{Hom}_{\mathrm{Ho}}(\mathcal{P}_C, \sigma_{\geq 1}\mathcal{P}_C) \leq 1.$$

i.e. *the set of morphisms of complexes up to homotopy from $\mathcal{P}_C$ to $\sigma_{\geq 1}\mathcal{P}_C$ is at most one dimensional.*

Proof. If $r = |C| = 0$, then $\sigma_{\geq 1}\mathcal{P}_C = 0$ and the space of morphisms in question is 0 dimensional. Assume that r is strictly bigger than 0. The setting of the lemma can be illustrated by the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus\limits_{s,t\in C} P_{s\wedge t} \arrow[r] \arrow[d] & \bigoplus\limits_{s\in C} P_S \arrow[r] \arrow[d] \arrow[dl, dashed, red, "0"] & P_{\hat{1}} \arrow[d] \arrow[dl, dashed, red, "0"] \\
\cdots \arrow[r] & \bigoplus\limits_{s,t\in C} P_{s\wedge t} \arrow[r] & \bigoplus\limits_{s\in C} P_S \arrow[r] & 0
\end{tikzcd}$$

The antichain C is inclusive so the projective indecomposables in degree i are associated to elements which are either bigger than the ones in degree $i+1$ or cannot be compared with them. Hence there are no non zero maps of degree 1. We thus need to describe the maps between $\mathcal{P}_C$ and $\sigma_{\geq 1}\mathcal{P}_C$ in the category of complexes. For $k \in [\![1, r]\!]$ both complexes have the same components so morphisms of complexes are determined by morphisms of modules

$$\phi_k \in \operatorname{End}\Big(\bigoplus_{\substack{S \subseteq C, \\ |S|=k}} P_{\wedge S} \Big)$$

satisfying the relation

$$\phi_k \circ \partial_{k+1} = \partial_{k+1} \circ \phi_{k+1}. \tag{3.7}$$

Since C is a strong antichain, the elements $\wedge S$ with a fixed cardinal cannot be compared. Hence an endomorphism ϕ_k of this module is of the form

$$\phi_k = \bigoplus_{|S|=k} \lambda_S \cdot id_{P_{\wedge S}} \tag{3.8}$$

with $\lambda_S \in \Bbbk$. The goal is to show that for $1 \leq k \leq r$ we have $\phi_k = \lambda_C \cdot id$. In other words, for all S, S' subsets of C, $\lambda_S = \lambda_{S'}$. We proceed by downward induction on k starting with $k = r$. In that case, we already have $\phi_r = \lambda_C id_{P_{\wedge C}}$ as there is only one projective indecomposable summand in degree r. If $r = 1$, we are done. Otherwise take $1 \leq k < r$ and assume $\phi_{k+1} = \lambda_C \cdot id$. We now put together equations (1.6), (3.7) and (3.8). On the one hand we have

$$\partial_{k+1} \circ \phi_{k+1} = \lambda_C \cdot \partial_{k+1}$$

Evaluating at $e_{\wedge S} \in P_{\wedge S} \subseteq \bigoplus_{|S|=k+1} P_{\wedge S}$.

$$\partial_{k+1} \circ \phi_{k+1}(e_{\wedge S}) = \lambda_C \cdot \sum_{s \in S} (-1)^{|s|_S} \cdot e_{\wedge(S \setminus \{s\})}. \tag{3.9}$$

On the other hand we have

$$\phi_k \circ \partial_{k+1}(e_{\wedge S}) = \sum_{s \in S} (-1)^{|s|_S} \lambda_{S \setminus \{s\}} \cdot e_{\wedge(S \setminus \{s\})}.$$

Because the $e_{\wedge(S \setminus \{s\})}$ are linearly independent we get

$$\lambda_{(S \setminus \{s\})} = \lambda_C$$

for all $S \subset C$ of cardinal $k+1$ and $s \in S$. Noticing that any subset of C of cardinal $k < r$ can be expressed as $S \setminus \{s\}$ for some S and some s, this finishes the proof. $\square$

Going back to the diamond (Figure 3.3), the following example illustrates how the result fails when the strong antichain condition is dropped

Example 3.4.5. The following morphism of complexes is not homotopic to any morphism of the form described in the previous lemma

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "1"] & P_0^3 \arrow[r] \arrow[d, "M"] & P_{c_1} \oplus P_{c_2} \oplus P_{c_3} \arrow[r] \arrow[d, "0"] & P_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P_0 \arrow[r] & P_0^3 \arrow[r] & P_{c_1} \oplus P_{c_2} \oplus P_{c_3} \arrow[r] & 0 & \end{tikzcd} \tag{3.10}$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

3.5 Main result

Theorem 3.5.1. *Let L be a finite lattice, let d and l be integers. Suppose there exists a family of antichains $(C_\alpha)_{\alpha \in L}$ of L such that for all $\alpha \in L$, the following assumptions hold.*

(i) The antichain C_α is below α.

(ii) The module M_C^α is non zero and there is an isomorphism

$$\mathbb{S}^l M_{C_\alpha}^\alpha \cong M_{C_\alpha}^\alpha[d] \tag{3.11}$$

in $D^b(\mathcal{A})$.

(iii) The antichain C_α is strong.

Then $D^b(\mathcal{A})$ is $\frac{d}{l}$-Calabi-Yau.

In practice, assumption (2) is the hardest to investigate. The proof relies on the following theorem.

Theorem 3.5.2. *[51, Theorem 3.1] Let X be a finite poset with a unique minimal or unique maximal element. If there are integers d and l such that $\mathbb{S}^l(P) \simeq P[d]$ for all projective indecomposable modules of X, then the category $D^b(\mathcal{A}_{\Bbbk}(X))$ is $\frac{d}{l}$-*fractionally Calabi-Yau.

We also use this classical *two-out-of three* lemma [61, Lemma 3.4.10].

Lemma 3.5.3. *Let $\mathcal{T}$ be a triangulated category with self equivalence T and let*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & TA_1 \arrow[d] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & TB_1 \end{tikzcd} \tag{3.12}$$

be a morphism between distinguished triangles. If two of the vertical morphisms are isomorphisms, then the third one is an isomorphism as well.

Notation 3.5.4. To simplify notation we now set $F = \mathbb{S}^l[-d]$. We will often refer to equation (3.11) as the Calabi-Yau property for a certain complex of $\mathcal{A}$-modules.

Proof of 3.5.1. By Theorem 3.5.2, it suffices to prove that the Serre functor satisfies the Calabi-Yau property on the projective indecomposable modules. We proceed by induction on the elements α of the lattice L. For the initial step notice that, by the first assumption of the claim, the indecomposable projective module associated to the minimum of the poset $P_{\hat{0}}[0] \simeq M^{\hat{0}}_{C_{\hat{0}}}[0]$ satisfies equation (3.11). Hence fix an element $\alpha > \hat{0}$ and assume as the induction hypothesis, that equation (3.11) is true for all $\alpha' < \alpha$. We will refer to this as the *outer induction hypothesis* (OIH). Write $C = C_\alpha$ for the antichain indexed by α and let $R = \mathcal{P}^\alpha_C$ be its associated projective resolution. If $r = |C| = 0$, then R is a projective module concentrated in degree 0 and there is nothing to prove by the second assumption in 3.5.1.
Step1. So assume $r > 0$ and consider the distinguished triangle

$$P_\alpha[0] \longrightarrow R \longrightarrow \sigma_{\geq 1}R \xrightarrow{f} P_\alpha[1] \tag{3.13}$$

induced by the truncation short exact sequence [54, Tag 0118]. A computation shows that the map f is the boundary map ∂_1 in degree 1 and zero in all other degrees. With **TR2** this triangle can be shifted to

$$R \longrightarrow \sigma_{\geq 1}R \longrightarrow P_\alpha[1] \longrightarrow R[1]. \tag{3.14}$$

Because F is triangulated, it sends this triangle to a triangle and we have the following diagram.

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\wr"] & \sigma_{\geq 1}R \arrow[r] \arrow[d, dashed, "\exists ?"] & P_\alpha[1] \arrow[r] & R[1] \arrow[d, "\wr"] \\ F(R) \arrow[r] & F(\sigma_{\geq 1}R) \arrow[r] & F(P_\alpha[1]) \arrow[r] & F(R)[1] \end{tikzcd}$$

If we can show that there exists an isomorphism following the dashed arrow and that the left square can be chosen to commute, we can apply **TR3** to complete the diagram followed by Lemma 3.5.3 to finish the proof.

Step 2. To construct an isomorphism

$$\sigma_{\geq 1} R \xrightarrow{\sim} F(\sigma_{\geq 1} R)$$

we show by downward induction on i that

$$\sigma_{\geq i} R \xrightarrow{\sim} F(\sigma_{\geq i} R)$$

for $i \in \{1, \ldots, r\}$. We refer to this as the *inner induction hypothesis*. Taking $i = r$, the initial step follows from the outer induction hypothesis as $\sigma_{\geq r} R \simeq P_{\wedge C}[r]$. Now, fix $1 < i < r$ and assume the property is true for i. To show that it is also true for $i - 1$, like in (3.13), consider the truncation triangle

$$\sigma_{\geq i} R \xrightarrow{\partial_i[i]} R^{i-1}[i] \longrightarrow \sigma_{\geq i-1} R[1] \longrightarrow \sigma_{\geq i} R[1]$$

shifted using **TR2**. Again, we want to conclude by using **TR3** and Lemma 3.5.3 so it suffices to show that we can choose the vertical isomorphisms of the following square such that it commutes.

$$\begin{tikzcd} \sigma_{\geq i} R \arrow[r, "\partial_i[i]"] \arrow[d, "f", "\wr"'] & R^{i-1}[i] \arrow[d, "g", "\wr"'] \\ F(\sigma_{\geq i} R) \arrow[r, "F(\partial_i[i])"] & F(R^{i-1}[i]) \end{tikzcd} \tag{3.15}$$

By the inner induction hypothesis we can choose an isomorphism f for the left vertical arrow and by the outer induction hypothesis we can choose an isomorphism g for the right vertical arrow.

Step 3. The isomorphism g can be rectified to make the square commutative. We compare the morphism ∂_i, with $g^{-1} \circ F(\partial_i[i]) \circ f$. Recall the isomorphism u between the morphism set in the homotopy and bounded derived category, provided that the source is a complex of projectives. The isomorphism is functorial in the target and is defined by sending a class of morphisms up to homotopy c to the morphism

$$\begin{tikzcd} & C & \\ P \arrow[ur, "c"] & & C \arrow[ul, "id"', "\sim"] \end{tikzcd}$$

in the derived category. This is the "fraction" $\frac{c}{1}$. In the homotopy category we want to compare the morphism ∂_i, which is already a morphism of complexes, with $u^{-1}(g^{-1} \circ F(\partial_i[i]) \circ f)$. By Lemma 3.4.1 there exists T an endomorphism of $\bigoplus_{|S|=i-1} P_{\wedge J}$ such that

$$(T \circ \partial_i)[i] = u^{-1}(g^{-1} \circ F(\partial_i[i]) \circ f). \tag{3.16}$$

Applying u again we get

$$\frac{T[i]}{1} \circ \frac{\partial_i[i]}{1} = g^{-1} \circ F(\partial_i[i]) \circ f.$$

From now on all the morphisms will be in the derived category but we omit the denominator of maps coming from the homotopy category as it is always equal to one.

We prove that the map T is an isomorphism. It has form $\bigoplus_S \lambda_S \cdot id_{P_{\wedge S}}$ because the antichain C is strong. It is enough to show that the projections of T onto the indecomposable summands of its target are non zero. Consider Figure 3.5.

$$\begin{tikzcd}
 & R^{i-1}[i] \arrow[r, "\pi_S[i]"] \arrow[d, "T[i]"] & P_{\wedge S}[i] \arrow[d, "\lambda_S \cdot id[i]"] & \text{Shifted module maps} \\
\sigma_{\geq i}R \arrow[ur, "\partial_i[i]"] \arrow[d, "f", "\text{IIH}"'] \arrow[r, phantom, "\text{by 3.4.1}"] & R^{i-1}[i] \arrow[r, "\pi_S[i]"] \arrow[d, "g", "\text{OIH}"'] & P_{\wedge S}[i] \arrow[d, "h \rightsquigarrow \tilde{h}", "\text{OIH}"'] & \dim \mathrm{Hom}(R^{i-1}[i], FP_{\wedge S}[i]) = 1 \\
F\sigma_{\geq i}R \arrow[r, "F(\partial_i[i])"] & FR^{i-1}[i] \arrow[r, "F(\pi_S[i])"] & FP_{\wedge S}[i] &
\end{tikzcd}$$

Figure 3.5: Maps at play and their projections on indecomposable summands

The pentagon on the left is commutative since

$$g \circ T[i] \circ \partial_i[i] = g \circ g^{-1} \circ F(\partial_i[i]) \circ f = F(\partial_i[i]) \circ f.$$

The top right square illustrates equation (3.5) so it commutes as well. For the bottom right square, the outer induction hypothesis gives the isomorphism h. Next, F is fully faithful so it induces an injective linear map on hom spaces. Hence $F(\pi_S[i])$ is non zero. Finally, g is an isomorphism so the composition $F(\pi_{\wedge S}[i]) \circ g$ is non zero too. Because the antichain is strong,

$$\dim \mathrm{Hom}_{\mathcal{A}}(R^{i-1}, P_{\wedge S}) = 1$$

and because $P_{\wedge S}[i]$ and $FP_{\wedge S}[i]$ are isomorphic, we also have

$$\dim \mathrm{Hom}_{D^b}(R^{i-1}[i], FP_{\wedge S}[i]) = 1.$$

Hence it is possible to replace the map h by $\tilde{h} = \lambda \cdot h$ for some λ making it commute. Chasing around the diagram we thus compute

$$\lambda_S \cdot \tilde{h} \circ \pi_S[i] \circ \partial[i] = F(\pi_S \circ \partial_i[i]) \circ f.$$

By construction the projections of ∂_i are all non zero and $\tilde{h}$ and f are isomorphisms. Again, F is an equivalence of categories so the right hand side is non zero. Hence $\lambda_S \neq 0$ for all $S \subseteq C$ of cardinal i and T is an automorphism. Rearranging equation (3.16) we get

$$\partial_i[i] = T^{-1} \circ g^{-1} \circ F(\partial_i[i]) \circ f.$$

So replacing g by $g \circ T$ the square (3.15) becomes commutative. Applying Lemma 3.5.3 and the inner induction hypothesis we have

$$F(\sigma_{\geq 1} R) \cong \sigma_{\geq 1} R.$$

Step 4. To complete the outer induction choose such an isomorphism g, and an isomorphism $f : R \to F(R)$. Just like in step 3 above, we want to compare the map $p : R \to \sigma_{\geq 1} R$ with $g^{-1} \circ F(p) \circ f$ and, if needed, rectify the isomorphisms g and f to make the square commute. Lemma 3.4.4 implies that $g^{-1} \circ F(p) \circ f = \lambda \cdot p$. Because g and f are isomorphisms and F induces a linear bijection between hom spaces, λ is non zero and we can replace g by $\lambda \cdot g$. This concludes the proof. $\square$

Chapter 4

A family of Fractionally Calabi-Yau Posets

The goal of this chapter is to prove Theorem C. We use the combinatorial family of antichain modules introduced in [60] and show that it satisfies the conditions of Theorem 3.5.1. For the convenience of the reader and because it will be used heavily in the rest of the article, in Section 4.1 we recall the combinatorial framework of [60] with small modifications. In Section 4.2 we also reproduce some proofs of [60], changing the statements when needed to obtain results in the derived category.

4.1 Grids and their order ideals

Let m and n be positive integers and $G_{m,n}$ be the product of two total orders of size m and n respectively. Recall from Example 2.1.33 the lattice $J_{m,n}$ of order ideals of $G_{m,n}$. We saw that $J_{m,n}$ is isomorphic as a lattice to non decreasing sequences of length m with values in $[\![0, n]\!]$ ordered by term wise integer comparison. These sequences are sometimes called *partitions* and written

$$(\lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r})$$

with $\sum_i \mu_i = m$, where μ_i encodes the multiplicity of the value λ_i and $\lambda_i \neq \lambda_j$ if $i \neq j$. These partitions can classically be counted as follows: choosing a partition amounts to putting m balls corresponding to coefficients $(a_1, \dots, a_m)$ into $n+1$ boxes, the possible values of the coefficients. Which also amounts to placing n sticks in $m+n$ possible slots. This means there are exactly $\binom{m+n}{m}$ partitions.

Bijections We now present other descriptions of this poset. The first two play a crucial role in the proof of Theorem C while the visualisations will be more helpful in the next section when describing hom spaces between certain objects. The first bijection that we introduce sends these non decreasing sequences of length m to increasing sequences of length m. Let

$$\mathcal{Z} = \{-m, \dots, -1, 0, 1, \dots, n\}$$

be a set of representatives of $\mathbb{Z}/(m+n+1)\mathbb{Z}$. A *configuration* C of $\mathcal{Z}$ is a subset of size m of $\mathcal{Z}$. We write it $C = \{c_1 < \cdots < c_m\}$ as an increasing sequence naively using the order relation on $\mathbb{Z}$. We write $C_{m,n}$ the set of configurations of length m on $\mathcal{Z}$. Choosing a configuration amounts to picking m distinct elements from a set of cardinal $m+n+1$ so the cardinality of $C_{m,n}$ is

$$\binom{m+n+1}{m}.$$

Given a partition α we can construct a configuration containing α's coefficients in its non negative side and encoding the multiplicities of α in its negative side. We write

$$x_i = \sum_{k=1}^{i} \mu_i$$

to record the index of the last occurence of the i^{th} coefficient. It will be called the *ending index* of the coefficient λ_i. We set $x_0 = 0$ as a convention. The index $x_{i-1}+1$ is the first occurence of λ_i and will be called the *starting index*. Think of the negative side as the indices of the elements of the sequence α but with a minus sign. Out of the many available options to encode the multiplicities, here are two that turn out to fit the problem perfectly:

- keep all negative elements except the opposite of the starting index of each coefficient. The resulting configuration is called the *left configuration* associated to α, and we denote by it L_α. The map sending α to L_α is denoted by ϕ_l;

- keep all negative elements except the opposite of the ending index of each coefficient. The resulting configuration is called the *right configuration* associated to α, and we denote it by R_α. The map sending α to R_α is denoted ϕ_r.

Example 4.1.1. Take $n = 7$, $m = 5$ and consider the partition $a = (0, 2, 3, 7, 7)$. We have $r = 4$ and

$$x_1 = 1, x_2 = 2, x_3 = 3 \text{ and } x_4 = 5.$$

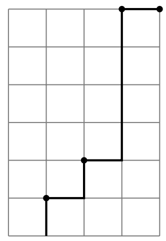
The associated left and right configurations are respectively

$$\{-5 < 0 < 2 < 3 < 7\} \text{ and } \{-4 < 0 < 2 < 3 < 7\}.$$

Proposition 4.1.2 ([60, Proposition 3.3])**.** *The maps ϕ_l and ϕ_r are injective.*

Proof. Partitions are entirely determined by their coefficients and multiplicities. These can be recovered from the positive elements of a configuration and its negative gaps, *i.e.* missing values, respectively. □

They are not surjective as $\phi_l(\alpha)$ cannot contain -1 and $\phi_r(\alpha)$ cannot contain $-m$.



Visualisation One can also think of partitions as paths in an $m \times n$ grid as depicted in the figure for the partition $(0, 2, 3, 7, 7)$. For a non decreasing sequence $(a_1, \ldots, a_m)$ the path is obtained by putting a dot at height a_i in the i^{th} column from the left and then take the minimal path going through this dots. If $a_i = 0$ put no dot. To visualise configurations consider a table with $m+n+1$ columns and put a dot, or a bead in the columns corresponding to the elements of the configuration at hand. We call this the *abacus* associated to the configuration.

Example 4.1.3. The abacus associated with the right configuration of the partition a from Example 4.1.1 is as follows

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
	•				•		•	•				•.

Note that we used the map ϕ_r for this configuration.

Each bead of the *negative side* adds one to the multiplicity of one of the coefficients. If we are using the map ϕ_r to obtain the configuration, the bead sitting in the $-k$ column, is associated to the k^{th} bead to its right. Indeed, gaps in the negative side indicate changes of coefficients. If the columns -1 to $-k+1$ are empty then exactly $(k-1)^{th}$ coefficients end before the k^{th} element of the sequence and the claim holds. Any extra bead between -1 and $-k+1$ removes one such gap. If there are s such beads in the negative side, the coefficient we are looking for corresponds to the $(k-s)^{th}$ bead in the positive side. Note that s is at most $k-1$, so the procedure always yields a bead in the positive side.

Example 4.1.4. Figure 4.1 illustrates this idea on the abacus of Example 4.1.3. Its only negative bead can be connected to the coefficient $\lambda_4 = 7$. It is the only coefficient with multiplicity 2 in the partition a.

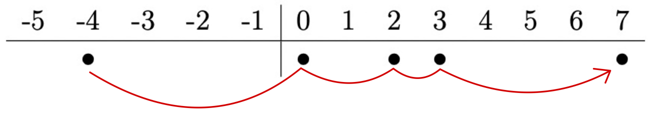


Figure 4.1: Illustrating the partial inverse of ϕ_r

Similarly, if the map ϕ_l was used, negative beads can be associated to their positive counterparts by skipping $k-1$ beads to the left. These two procedures provide a way to recover a partition from an abacus by associating the correct multiplicity to each value indicated in the positive side.

4.1.1 A Family of antichain modules

To apply the machinery presented in Chapter 3, we identify a well behaved family of antichains. Luckily, the poset of partitions is remarkably well suited for the matter: to create an antichains below a certain element α it suffices to identify transformations on α whose respective support cannot be compared. The family we discuss was discovered by Yıldırım and used to prove that the Coxeter transformation of the lattice $J_{m,n}$ is periodic. The rest of the thesis will focus on these antichains and their associated modules. We introduce some extra combinatorial data before discussing the antichains themselves, their associated modules and projective resolutions.

Enhancements We consider several antichains below x for each $x \in J_{m,n}$. These antichains can be encoded as *decorations* or *enhancements* of x.

Definition 4.1.5. A *right enhanced partition* is a sequence

$$(\lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$$

where multiplicities $\mu_1, \dots, \mu_{r+1}$ sum to m. Note that we allow $\mu_{r+1} = 0$. In that case, we say the partition is *plain*. If $\mu_{r+1} \neq 0$ we say the partition is *strictly* enhanced. We denote by $E^R_{m,n}$ the set of right enhanced partitions. Similarly a *left enhanced partition* is a sequence

$$(0^{\mu_0} | \lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r}).$$

Example 4.1.6. The right enhanced partition $\beta = (0, 1, 3^2, 4, 5|5^2)$ can be represented as the path in Figure (4.2) where the enhancement is recorded with a short line. Here we put the enhanced values in red for emphasis.

The set of left enhanced partitions will be written $E^L_{m,n}$. As before, a left partition with $\mu_0 = 0$ is called plain, otherwise it is strictly enhanced. Like for plain partitions, we count

$$\binom{m+n+1}{m}$$

right enhanced partitions as well as left enhanced ones. The map ϕ_r naturally extends to right enhanced partitions without any change, making it a bijection. By "without any change" we mean that, as for plain partitions, we remove the values $-x_1$ to $-x_r$ in the negative side and do not remove $-x_{r+1} = -m$. Similarly, the map ϕ_l extends to left enhanced partitions, not removing -1 from the negative side when $\mu_0 > 0$.

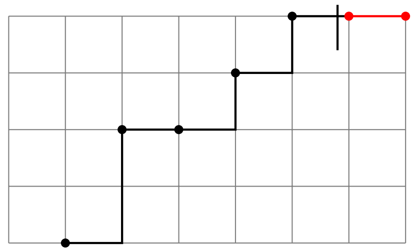


Figure 4.2: Enhanced partitions in grids

Example 4.1.7. Consider enhanced versions of the partition a from example 4.1.1. The right configuration associated to the right enhanced partitions $(0, 2, 3, 7 \,|\, 7)$ and $(0, 2, 3 \,|\, 7, 7)$ are

$$\{-5 < 0 < 2 < 3 < 7\} \text{ and } \{-5 < -4 < 0 < 2 < 3\}$$

respectively. The left configuration associated to the left enhanced partition $(0 \,|\, 2, 3, 7, 7)$ is

$$\{-5 < -1 < 2 < 3 < 7\}.$$

These configurations are represented in an abacus, in the order they were mentioned as follows:

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
•					•		•	•				•
•	•				•		•	•				
•				•			•	•				•

Corresponding antichains Our antichains are obtained by modifying the coefficients and leaving multiplicities unchanged. For a right enhanced partition $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ define the *mutable coefficients* to be $S_\alpha = \{\epsilon, \ldots, r\}$ the indices corresponding to nonzero coefficients. The number ϵ is either 1 or 2. Please remark that this excludes the coefficients beyond the enhancement bar.

Definition 4.1.8. Let $\alpha = (\lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ be a right enhanced partition. For any subset J of S_α define a new partition $q_J(\alpha) = ((\lambda_1')^{\mu_1}, \dots, (\lambda_r')^{\mu_r} | n^{\mu_{r+1}})$ by

$$\lambda_i' = \begin{cases} \lambda_i - 1 & \text{if } i \in J, \\ \lambda_i & \text{otherwise.} \end{cases}$$

Consider now the set

$$C_\alpha = \{q_i(\alpha) | i \in S_\alpha\}. \tag{4.1}$$

Because $q_i(\alpha)$ and $q_j(\alpha)$ differ from α at different indices, their associated plain partitions form an antichain and we denote $\mathcal{P}_\alpha$ the perfect complex associated to it.

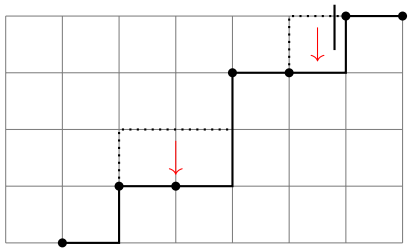


Figure 4.3: Illustration of q_K

We have opted to define $q_i(\alpha)$ as enhanced partitions as these transformations will also parametrise extensions between the objects $\mathcal{P}_\alpha$. This is the topic of the next section. However the antichains associated with this construction are made up of elements of the poset, which are plain partitions.

Example 4.1.9. Consider the right enhanced partition $\beta = (0, 1, 3^2, 4, 5|5^2)$ from example 4.1.6. Then we have $S_\beta = \{2, 3, 4, 5\}$. Picking the subset $K = \{3, 5\}$ of S_β yields $q_K(\beta) = (0, 1, 2^2, 4^2|5^2)$. See Figure (4.3).

Proposition 4.1.10. *The antichain C_α is a boolean antichain below α.*

Proof. First we check that the antichain is *strong*. We take $I, J \subseteq S_\alpha$ such that $|I| = |J| > 0$ and $I \neq J$. We also take $i \in I \setminus J$ and $j \in J \setminus I$. Then it holds that

$$\lambda_i - 1 = q_I(\alpha)_{x_i} < q_J(\alpha)_{x_i} = \lambda_i$$

and symmetrically at the index x_j. Hence $q_I(\alpha)$ and $q_J(\alpha)$ cannot be compared and the antichain is strong. Now we check that the antichain is *intersective*. Recall that the join of two partitions is the termwise maximum. We take $I, J \subset S_\alpha$ to be non empty. We write

$$\gamma = q_I(\alpha) \vee q_J(\alpha).$$

For each $i \in S_\alpha$, and every $k \in]x_{i-1}, x_i]$ we have $\gamma_k = \lambda_i - 1$ if $k \in I \cap J$ and $\gamma_k = \lambda_i$ otherwise. Hence $\gamma = q_{I \cap J}(\alpha)$. ∎

Associated Intervals

Proposition 4.1.11 ([60, Proposition 2.13])**.** *Let α be a right enhanced partition. Then $\mathcal{P}_\alpha$ is a projective resolution of the interval $[f(\alpha), \alpha]$ where the function f is defined by*

$$f : (\lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r} | n^{\mu_{r+1}}) \mapsto (0^{\mu_1 - 1} | \lambda_1^{\mu_2}, \dots, \lambda_r^{\mu_{r+1}+1}). \tag{4.2}$$

It is more convenient to define the image of a right enhanced partition by the function f to be a left enhanced partition. However, the interval $[f(\alpha), \alpha]$ is an interval of $J_{m,n}$, *i.e* an interval with bounds the corresponding plain partitions.

Proof. An element β of the lattice is in the support of the antichain module associated to $\mathcal{P}_\alpha$ if and only if $\beta \le \alpha$ and for all $i \in S_\alpha$, we have $\beta \not\le q_i(\alpha)$. Because the partition $q_i(\alpha)$ differs from α only between the indices $x_{i-1}+1$ and x_i, there exists $k \in [x_{i-1}+1, x_i]$ such that $b_k = a_k = \lambda_i$. The sequence β is increasing so this is equivalent to $b_{x_i} = \lambda_i$ for all $i \in S_\alpha$. The partition $f(\alpha)$ satisfies these conditions. For any other β satisfying them, for $k \in [x_{i-1}+1, x_i]$, we have $f(\alpha)_k = \lambda_{i-1} = b_{x_{i-1}} \le b_k$, hence $\beta \in [f(\alpha), \alpha]$. $\square$

This proof uses the fact that the support of the antichain module associated to $\mathcal{P}_\alpha$ is the set

$$\{\beta \le \alpha | \forall i \in S_\alpha, \beta_{x_i} = \lambda_i\}. \tag{4.3}$$

This highlights the role of the indices x_i for $i \in S_\alpha$. These indices will serve as comparison points between our partitions in many proofs to follow.

Example 4.1.12. Consider the partition

$$\beta = (0, 1, 3^2, 4, 5|5^2)$$

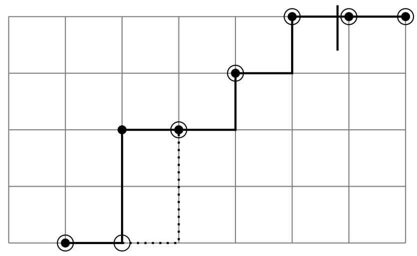


Figure 4.4: partitions β and $f(\beta)$

from example 4.1.6. Then $f(\beta) = (|0, 1^2, 3^1, 4, 5^3)$ with the corresponding path dotted in figure (4.4). Please note how the value of β and $f(\beta)$ match at the ending indices $1, 3, 4$ and 5 and how the values of $f(\beta)$ are minimal given those constraints.

4.2 Yıldırım's theorem

The following key result is a categorified version of [60, Propoposition 4.2].

Proposition 4.2.1. *Let α be a right enhanced partition. Then*

$$\mathbb{S}^{m+n+1}(\mathcal{P}_\alpha) \cong \mathcal{P}_\alpha[mn].$$

First we describe the action of the Serre functor on the object $\mathcal{P}_\alpha$. Recall that it sends the projective indecomposable P_α to the injective indecomposable I_α.

Definition 4.2.2. We call $\mathcal{I}_\alpha$ the image of the projective resolution $\mathcal{P}_\alpha$ under the Serre functor. Because $\mathcal{P}_\alpha$ is a complex of projective modules, we get $\mathcal{I}_\alpha$ by tensoring the components of P_α by $D\mathcal{A}$ which gives us

$$\mathcal{I}_\alpha : 0 \to I_{q_{S_\alpha}(\alpha)} \xrightarrow{\partial_r} \bigoplus_{\substack{J\subseteq S_\alpha,\\ |J|=r-1}} I_{q_J(\alpha)} \to \cdots \to \bigoplus_{\substack{J\subseteq S_\alpha,\\ |J|=r-k}} I_{q_J(\alpha)} \xrightarrow{\partial_{r-k}} \cdots \to I_\alpha \to 0. \tag{4.4}$$

Just like for $\mathcal{P}_\alpha$, we can show that $\mathcal{I}_\alpha$ is an injective resolution. Its homology is concentrated in degree $|S_\alpha|$. To describe this module, define the map

$$g : (0^{\alpha_0}|\lambda_1^{\alpha_1}, \ldots, \lambda_r^{\alpha_r}) \mapsto (\lambda_1^{\alpha_0+1}, \lambda_2^{\alpha_1}, \ldots, \lambda_r^{\alpha_{r-1}}|n^{\alpha_r-1}) \tag{4.5}$$

which is a counterpart to the map f, and set the following enhancement on the partition $q_{S_\alpha}(\alpha)$

$$\begin{cases} (0^{\alpha_1}|(\lambda_2-1)^{\alpha_2}, \ldots, (\lambda_r-1)^{\alpha_r}, n^{\alpha_{r+1}}) & \text{if } \lambda_1 = 0 \\ (|(\lambda_1-1)^{\alpha_1}, \ldots, (\lambda_r-1)^{\alpha_r}, n^{\alpha_{r+1}}) & \text{otherwise.} \end{cases} \tag{4.6}$$

We denote δ the map sending α to this enhancement of $q_{S_\alpha}(\alpha)$. Then the support of the homology of $\mathcal{I}_\alpha$ in degree $|S_\alpha|$ is the interval

$$[\delta(\alpha), g \circ \delta(\alpha)]. \tag{4.7}$$

The map delta amounts to representing the corresponding configuration in an abacus using ϕ_r, shift all its beads one step to the left and interpret the configuration using ϕ_l. If we denote by T_{-1} the shifting of the beads by one to the left, we can write this as $\delta = \phi_l^{-1} \circ T_{-1} \circ \phi_r$. The maps f and g are further related by the following lemma.

Lemma 4.2.3. *The functions f and g are inverse of one another.*

Proof. To see that it suffices to compute both compositions

$$
\begin{aligned}
g \circ f : (\lambda_1^{\alpha_1}, \dots, \lambda_r^{\alpha_r} | n^{\alpha_{r+1}}) &\mapsto (0^{\alpha_1 - 1} |, \lambda_1^{\alpha_2}, \dots, \lambda_{r-1}^{\alpha_r}, \lambda_r^{\alpha_{r+1}+1}) \\
&\mapsto (\lambda^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_r^{\alpha_r} | n^{\alpha_{r+1}+1-1})
\end{aligned}
$$

and, almost dually we have:

$$
\begin{aligned}
f \circ g : (0^{\alpha_0} | \lambda_1, \dots, \lambda_r^{\alpha_r}) &\mapsto (\lambda_1^{\alpha_0+1}, \lambda_2^{\alpha_1}, \dots, \lambda_r^{\alpha_{r-1}} | n^{\alpha_r - 1}) \\
&\mapsto (0^{\alpha_0+1-1} | \lambda_1, \dots, \lambda_{r-1}^{\alpha_{r-1}}, \lambda_r^{\alpha_r - 1 + 1}).
\end{aligned}
$$

□

The key for Yıldırım's theorem is the following proposition which mirrors the combinatorics of Proposition 4.2 in [60] but with slightly different combinatorial objects.

Proposition 4.2.4. *On the abacus, the map f is a shift to the left of the negative beads or, to put it simply $f(\alpha) = \phi_l^{-1} \circ \phi_r(\alpha)$. Conversely g is the reinterpretation of the configuration using ϕ_r instead of ϕ_l* i.e. *$g(\alpha) = \phi_r^{-1} \circ \phi_l(\alpha)$.*

Proof. Both α and $f(\alpha)$ have the same *plain* coefficients $\lambda_1, \dots, \lambda_r$ of α hence their respective abaci have the same positive beads regardless of the map used. To see that they coincide in the negative side notice that

$$x_{i-1}^{f(\alpha)} + 1 = x_i^{\alpha}$$

for $i \in S_\alpha$. The result on g follows from Lemma 4.2.3 □

More importantly we can now show that Yıldırım's family of intervals is stable under the Serre functor. Write $\tilde{f} = g \circ \delta$. Then we have

Proposition 4.2.5. *Let α be a left enhanced partition. Then $\mathbb{S}(\mathcal{P}_\alpha) \simeq \mathcal{P}_{\tilde{f}(\alpha)}[|S_\alpha|]$.*

Proof. Recall that $\mathbb{S}(\mathcal{P}_\alpha)$ is isomorphic to the interval $[q_{S_\alpha}(\alpha), g(q_{S_\alpha}(\alpha))]$ shifted by $|S_\alpha|$. But because f and g are inverse of one another this is the interval

$$[f(\tilde{f}(\alpha)), \tilde{f}(\alpha)]$$

which itself is isomorphic to $\mathcal{P}_{\tilde{f}(\alpha)}$ by Equation 4.2. This gives us the result. □

Hence describing the action of the Serre functor on the Yıldırım modules amounts to describing the action of $\tilde{f}$ on abaci. This turns out to be quite simple

Lemma 4.2.6. *The map $\tilde{f}$ acts on right abaci associated to right enhanced partitions as a shift to the left*

Justification. This is clear form the description in terms of abaci of g in Proposition 4.2.4 and of δ below equation (4.7). □

Example 4.2.7. Recall the partition $\beta = (0, 1, 3^2, 4, 5|5^2)$ from Example 4.1.6. We computed $f(\beta) = (|0, 1^2, 3^1, 4, 5^3)$ in Example 4.1.12. Now, following equations (4.6) and (4.5) respectively, we compute $\delta(\beta) = (0|0, 2^2, 3, 4, 5^2)$ and $\tilde{f}(\beta) = g \circ \delta(\beta) = (0^2, 2, 3^2, 4, 5|5)$. The right configuration associated to this partition is $\{-8, -4, -1, 0, 2, 3, 4, 5\}$ while the one associated to β is $\{-8, -7, -3, 0, 1, 3, 4, 5\}$. The following abacus represents β and $\tilde{f}(\beta)$ and illustrates Lemma 4.2.6.

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
•	•				•			•	•		•	•	•
•				•			•	•		•	•	•	•

Proof of 4.2.1. The previous Lemma implies that $\tilde{f}^{n+m+1}$ is the identity on configurations hence on partitions meaning that $\mathbb{S}^{m+n+1}(\mathcal{P}_\alpha)$ is isomorphic to $\mathcal{P}_\alpha$ with a certain shift. To compute that shift, recall that applying the Serre functor to $\mathcal{P}_\alpha$ shifts the resolution by $|S_\alpha|$ to the left so the total shift is:

$$\sum_{i=1}^{m+n+1} |S_{\tilde{f}^i(\alpha)}|.$$

But $|S_{\alpha'}|$ is the number of non zero beads in the positive side of the right abacus associated to α'. When applying $\tilde{f}$ to α a total of $n + m + 1$ times, each bead will be in the positive side exactly n times. Because there are m beads, this means the shift is nm. □

Proof of Theorem C. Let us check that the family

$$\{\mathcal{P}_\alpha | \alpha \text{ enhanced } (m, n)\text{-partition}\}$$

satisfies the three conditions of Theorem 3.5.1. First, no matter the enhancement of a partition α, the projective cover of the interval $[f(\alpha), \alpha]$ is the indecomposable projective P_α. Theorem 4.2.1 gives the second condition. The third condition follows from Proposition 4.1.10, and this concludes the proof. □

Chapter 5

Tilting to higher Auslander algebras of type A

The goal of this chapter is to prove Theorem E. We start by describing morphisms and extensions between the $\mathcal{P}_\alpha$ antichain modules introduced in the previous chapter. After discussing relations between the morphisms, we construct a tilting complex between $J_{m,n}$ and A_{m+1}^{n-1}. In doing so we make explicit an isomorphism between the higher Auslander algebra of type A_s^d and the quadratic dual of A_{d+2}^{s-2}.

5.1 Describing Homspaces

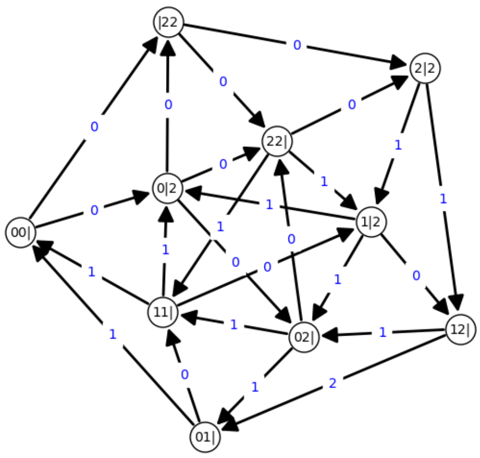


Figure 5.1: Graph of the $\mathcal{Y}_{2,2}$

We define the category $\mathcal{Y}_{m,n}$ to be the full subcategory of $D^b(J_{m,n})$ whose objects are the complexes $\mathcal{P}_\alpha$ and all their shifts. The goal of this section is to describe the morphisms of this category and to identify irreducible morphisms. These include morphisms of modules whose homology is concentrated in degree zero as well as morphisms between shifted objects which are in fact extensions. Given a morphism in $\mathcal{Y}_{m,n}$, we will first factor it through an extension of the same degree but which we can easily describe using our antichains. This factorisation yields a degree zero map. We then decompose further these two components. Starting the process of our factorization with the extension yields a form of asymmetry between morphisms and extensions. It is an artefact of this proof which will be smoothed out in the next section.

As a visual help and source of examples, see Figure 5.1 for the graph of $\mathcal{Y}_{2,2}$. The labels on the arrows indicate the degree in which the morphisms are concentrated. Note that relations do not appear on the figure: many compositions of arrows are in fact zero. Let $\alpha = (\lambda_1^{\mu_1}, \dots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ and $\beta = (l_1^{m_1}, \dots, l_s^{m_s} | n^{m_{s+1}})$ be partitions. In this section we will consistently use the following convention for the ending indices of the coefficients of α and β

$$x_i = \sum_{k \leq i} \mu_k \text{ and } y_j = \sum_{k \leq j} m_k$$

to distinguish data from the two partitions as well as the indices that we might be looking at any given moment. Recall from Theorem 3.3.4 that there exists at most one integer p such that

$$\dim \operatorname{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta[p]) \neq 0.$$

If it exists then $\dim \operatorname{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta[p]) = 1$. This is the case *if and only if* there exists a unique subset J of S_α such that $q_J(\alpha) \in [f(\beta), \beta]$. In that case, $p = |J|$. When $p = 0$, we can give different characterisations of this property.

Proposition 5.1.1. *The following are equivalent.*

(i) There exists a nonzero morphism $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta$.

(ii) The inequalities $f(\alpha) \leq f(\beta) \leq \alpha \leq \beta$ hold.

(iii) The partition α is in $[f(\beta), \beta]$ and for all non empty $J \subseteq S_\alpha$ we have

$$q_J(\alpha) \notin [f(\beta), \beta].$$

(iv) The partition α is in $[f(\beta), \beta]$ and $\{\lambda_i | i \in S_\alpha\} \subseteq \{l_j | j \in S_\beta\}$.

(v) For all $j \in S_\beta$ there exists $i \in S_\alpha \cup \{r+1\}$ such that $\lambda_i = l_j$, $x_{i-1} < y_j \leq x_i < y_{j+1}$ and $\{\lambda_i | i \in S_\alpha\} \subseteq \{l_j | j \in S_\beta\}$.

Proof. **(i) ⇔ (ii)** is the characterisation of morphisms between intervals recalled in equation (1.5).

(i) ⇔ (iii) follows from Theorem 3.3.4.

(iv) ⇔ (v) Reformulate (iv) using the ideas of the proof of Proposition 4.1.11 which we reproduce here. Recall that the interval $[f(\beta), \beta]$ is the set of partitions $\{\gamma | \gamma \leq$

β and for all $j \in S_\beta, \gamma \not\leq q_j(\beta)\}$. Because the partition $q_j(\beta)$ differs from β between indices $y_{j-1}+1$ and y_j, this is equivalent to the existence of $k \in [y_{j-1}+1, y_j]$ such that $l_j - 1 < a_k = b_k = l_j$. Equivalently, there exists $i \in S_\alpha$ such that $a_k = \lambda_i = l_j = b_k$. The equality $a_k = b_k$ is equivalent to saying there is an overlap of the occurrences of the value $l_j = \lambda_i$ in β and α. This translates into the interlacing $x_{i-1} < y_j \leq x_i < y_{j+1}$.
To complete the proof we argue that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (iv) Assume that for all $i \in S_\alpha$, we have $q_i(\alpha) \notin [f(\beta), \beta]$. Thus there exists $j \in S_\beta$ such that $(q_i(\alpha))_{y_j} < l_j$. Because α and $q_i(\alpha)$ differ only for indices between $x_{i-1}+1$ and x_i we conclude that $\lambda_i = l_j$. The inclusion $\{\lambda_i | i \in S_\alpha\} \subseteq \{l_j | j \in S_\beta\}$ follows.

(iv) $\Rightarrow$ (ii) Remember that in $f(\beta)$, the value l_j runs from y_j to $y_{j+1}-1$. Hence, the interlacing condition implies that $(f(\beta))_{x_i} = \lambda_i$ and $f(\beta) \in [f(\alpha), \alpha]$ because for all $i \in S_\alpha$, we have $q_i(\alpha) \not\leq f(\beta)$.

□

To describe some degree zero morphisms, we introduce new transformations on $J_{m,n}$.

Definition 5.1.2. Let $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ be an enhanced partition. Let i be in S_α. If $\mu_i > 1$ define the partition $p_i(\alpha) = (\lambda_1^{m_1}, \ldots, \lambda_r^{m_r} | n^{m_{r+1}})$ with multiplicities

$$m_j = \begin{cases} \mu_j - 1 & \text{if } j = i, \\ \mu_j + 1 & \text{if } j = i+1, \\ \mu_j & \text{otherwise.} \end{cases}$$

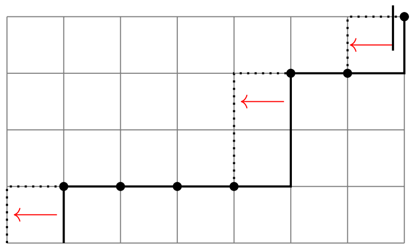


Figure 5.2: p_1, p_2 and p_3 on $\alpha = (0, 2^4, 4^2|5)$

There is a special case when $i = 1$, $\mu_1 = 1$ and $\lambda_1 = 0$: we still define the partition p_1

$$p_1(\alpha) = p_1(0^1, \lambda_2^{\mu_2}, \ldots, \lambda_r^{\mu_r}) = (\lambda_2^{\mu_2+1}, \lambda_3^{\mu_3}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}}).$$

The resulting partition has one coefficient less than α which reduces by one the index of the coefficients which remain in the partition. This will lead to technicalities in some proofs. Finally, when $\lambda_r < n$ and $\mu_{r+1} > 0$, we denote $p_\star(\alpha)$, the partition

$$(\lambda_1^{\mu_1}, \ldots \lambda_r^{\mu_r}, n^{\mu_{r+1}}|)$$

which has one more mutable coefficient than α. In these three cases we say that p_i is *well defined*.

Where q_j changed the values of the partition, p_i acts on its multiplicities. Using the previous proposition and this new definition we list some notable degree zero morphisms.

Corollary 5.1.3. *Consider a right enhanced partition α. There exists a non zero morphism $\mathcal{P}_\alpha \to \mathcal{P}_\beta$ whenever $\beta = p_i(\alpha)$ and $i \in \{1, \dots, r\} \cup \{\star\}$ such that p_i is well defined.*

Proof. We first consider the case where $i \neq \star$ and $\mu_i > 1$. Notice that if p_i is well defined, then the ending indices of α and $p_i(\alpha)$ are the same except from $x_i^\alpha = x_i^{p_i(\alpha)} + 1$. Hence, knowing that $\mu_i > 1$ we have $\alpha_{x_j^\alpha - 1} = \lambda_i = p_i(\alpha)_{x_j^{p_i(\alpha)}}$ for all $j \in S_\alpha$. It follows that $\alpha \in [f(p_i(\alpha)), p_i(\alpha)]$ using the characterisation from equation (4.3). We also know from Proposition 4.1.11 that $f(p_i(\alpha))$ and $p_i(\alpha)$ have the same values at indices $x_i^{p_i(\alpha)}$ from which we can deduce that $f(p_i(\alpha)) \in [f(\alpha), \alpha]$ using the characterisation equation (4.3) again. We conclude with item (ii) from Proposition 5.1.1. The other two cases can be treated very similarly and are left to the reader. See Figure 5.3 for a visualisation of the intervals. $\square$

Notation 5.1.4. For a pair (α, β) as in Lemma 5.1.1, consider the composition of *canonical* maps

$$P_\alpha \xhookrightarrow{\iota_\alpha^\beta} P_\beta \twoheadrightarrow [f(\beta), \beta].$$

By Item (iii) of Lemma 5.1.1, for $j \in S_\alpha$ we have $q_j(\alpha) \notin [f(\beta), \beta]$. Hence the map above sends generators of N_C^α to zero and this factors uniquely through $\mathcal{P}_\alpha \cong P_\alpha / N_\alpha \cong [f(\alpha), \alpha]$ providing us with one instance of non zero morphism which we denote ${}^0u_i$. Because the hom space is one dimensional, any other non zero morphism is proportional to ${}^0u_i^\alpha$.

We will later see that degree zero morphisms occur from $\mathcal{P}_\alpha$ to $\mathcal{P}_\beta$ if and only if there exists a sequence $I = (i_1, \dots, i_k)$ of $i \in \{1, \dots, r\} \cup \{\star\}$ satisfying $\beta = p_{i_k} \circ \dots \circ p_{i_1}(\alpha)$ such that the intermediate transformations are well defined. In that case we write $\beta = p_I(\alpha)$. Moreover, the ${}^0u_i$ will be shown to correspond to irreducible morphisms. Because of the combinatorial nature of the argument we will deal with this later and focus first on the extensions. The following definition characterizes the subsets J of S_α that will play a role in describing extensions.

Definition 5.1.5. A subset J of S_α is *allowed* when for all $i \in J$ such that $i - 1 \in S_\alpha \setminus J$, we have $\lambda_{i-1} < \lambda_i - 1$.

Example 5.1.6. Consider the partition $\alpha = (0, 1^2, 3, 4^2, 5|5)$ with $S_\alpha = \{2, 3, 4, 5\}$ and $\mu_{r+1} = 1$. Then the subset $J = \{2, 3, 4\}$ and $\{2\}$ are allowed while $\{4\}$ and $\{5\}$ are not.

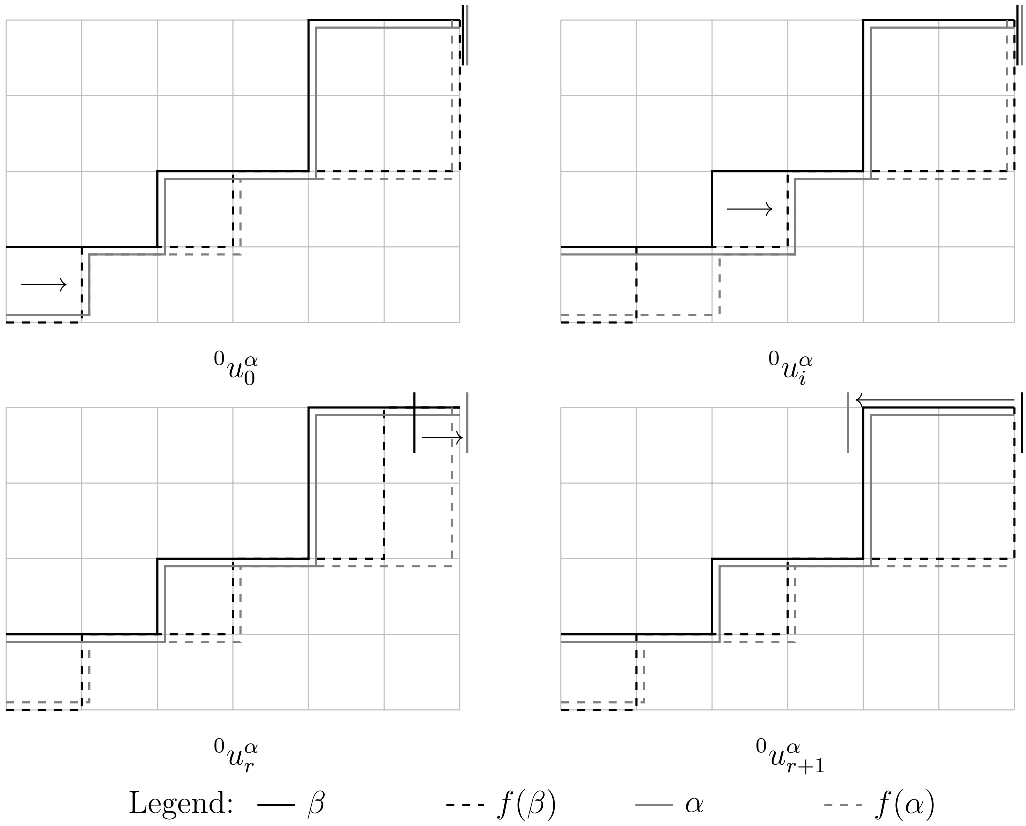


Figure 5.3: Illustration of the interpolation condition for certain degree zero morphisms.

The following lemma records some useful facts about allowed subsets and the combinatorics of the p_i transformations. The notation $J-1$ refers to the set $\{j-1|j \in J\}$.

Lemma 5.1.7. *Let J be a subset of S_α, let h, j be elements of S_α and let i, k be in $\{1, \dots, r\} \cup \{\star\}$. The following assertions hold.*

(i) If J is allowed, partitions α and $q_J(\alpha)$ have the same multiplicities for matching non zero values.

(ii) If J is allowed and $2 \notin J$ or $\lambda_2 > 1$, then for all $J \subseteq I \subseteq S_\alpha$ we have $q_I(\alpha) = q_{I \setminus J}(q_J(\alpha))$.

(iii) Keeping I and J as in Item (ii) but taking $\lambda_2 = 1$ and $2 \in J$ then $q_I(\alpha) = q_{(I\setminus J)-1}(q_J(\alpha))$.

(iv) If J is allowed, let $I \subseteq S_\alpha$ and $I' \subseteq S_{q_J(\alpha)} \setminus J$ be two sets of cardinal k and $k-|J|$ respectively. If $2 \notin J$ or $\lambda_2 > 1$ assume $I' \neq I \setminus J$. Otherwise assume $I' \neq (I \setminus J)-1$. Then the partitions $q_I(\alpha)$ and $q'_I(q_J(\alpha))$cannot be compared.

(v) *If p_i is well defined and $i \neq 1$ or $\mu_1 > 1$ then $q_J(p_i(\alpha)) > q_J(\alpha)$ and for any $J' \subseteq S_{p_i(\alpha)}$ different from J but of same cardinal, we have $q_{J'}(p_i(\alpha)) \not> q_J(\alpha)$.*

(vi) *If $i = 1$, $\mu_1 = 1$ and $\lambda_1 = 0$ then $q_{J-1}(p_i(\alpha)) > q_J(\alpha)$ and for any $J' \subseteq S_\beta$ different from $J - 1$ but of same cardinal, we have $q_{J'}(p_i(\alpha)) \not> q_J(\alpha)$.*

(vii) *If both h and j are allowed for α we have $q_h q_j(\alpha) = q_j q_h(\alpha)$.*

(viii) *If both p_i and p_k are well defined for α we have $p_i p_k(\alpha) = p_k p_i(\alpha)$.*

(ix) *If j is allowed for α, and $j \neq 2$ or $\lambda_2 > 1$, and i is well defined for α and $i \neq 1$ or $\mu_1 > 1$, then $p_i q_j(\alpha) = q_j p_i(\alpha)$.*

Proof. (i) From Definition 4.1.8, the partition $q_J(\alpha)$ can be written as

$$(\lambda'_1, \dots, \lambda'_1, \lambda'_2, \dots, \lambda'_2, \dots, \lambda'_r, \dots, \lambda'_r | n, \dots, n)$$

where the coefficient λ'_i appears μ_i times. In general we do not know if the λ'_j coefficients are all distinct. Because J is allowed, for all $j \in J$ we either have $j - 1 \in J$ and $\lambda'_j = \lambda_j - 1 > \lambda'_{j-1} = \lambda_{j-1} - 1$, or, $j - 1 \notin J$ in which case, if $j \neq 1$, $\lambda'_j = \lambda_j - 1 > \lambda'_{j-1} = \lambda_{j-1}$. For $i \notin J$, then we have $\lambda'_i = \lambda_i > \lambda_{i-1} \geq \lambda'_{i-1}$ regardless of whether $i - 1 \in S_\alpha$ or not. Hence all the non zero values λ'_i appear exactly μ_i times just like their counterpart in α. When $\lambda_2 = 1$ and $2 \in J$, then 0 appears $\mu_1 + \mu_2$ times in $q_i(\alpha)$

(ii) Let i be in S_α and $k \in [x_{i-1} + 1, x_i]$. Then

$$(q_{I \setminus J}(q_J(\alpha)))_k = \begin{cases} \lambda_i - 1 & \text{if } i \in I \setminus J \text{ or } i \in J \text{ } i.e. \text{ if } i \in I \\ \lambda_i & \text{otherwise} \end{cases}$$

which corresponds exactly to $q_I(\alpha)$.

(iii) When $\lambda_2 = 1$ and $2 \in J$, the first non zero coefficient has values λ_3 or $\lambda_3 - 1 > 0$. It is the second coefficient in $q_J(\alpha)$. More generally, when J is allowed, the coefficient present between indices $x_{i-1}+1$ and x_i is λ_i or $\lambda_i - 1$ but is the $(i-1)^{th}$ coefficient in $q_J(\alpha)$. Hence, the subset I' of $S_{q_J(\alpha)}$ which satisfies $q_{I'}(q_J(\alpha)) = q_I(\alpha)$ is $(I \setminus J) - 1$.

(v) The transformation p_i is well defined and $i \neq 1$ or $\mu_i > 1$, so α and $p_i(\alpha)$ have the same coefficients $\lambda_1, \dots, \lambda_r$. Hence $q_J(\alpha)$ and $q_J(p_i(\alpha))$ have the same coefficients $\lambda'_1, \dots, \lambda'_s$. They differ only at the index x_i where $q_J(p_i(\alpha))_{x_i} = \lambda'_{i+1} > \lambda'_i = q_J(\alpha)_{x_i}$.

Hence $q_J(p_i(\alpha)) \geq q_J(\alpha)$. Now take $J' \subseteq S_{p_i(\alpha)}$ of same cardinal as J but different from it and let j be in $J' \setminus J$. The partitions $q_{J'}(p_i(\alpha))$ has value $\lambda_j - 1$ between the indices $x'_{j-1}+1$ and x'_j, where the values $x'_{j-1}+1$ and x'_j varie depending on whether $i = j-1$, $i = j$ or neither. It is easy to check that in all three cases we have $x'_{j-1}+1 \leq x_j \leq x'_j$. At the same time partition $q_J(\alpha)$ has value λ_j at index x_j. Hence $q_{J'}(p_i(\alpha)) \ngeq q_J(\alpha)$.

The proofs of the remaining items are left for the reader hoping the ones we gave provide enough intuition on how the combinatorics of partitions will be discussed in the rest of the thesis. □

Proposition 5.1.8. *There is a nonzero morphism from $\mathcal{P}_\alpha$ to $\mathcal{P}_{q_J(\alpha)}[i]$ in the derived category if and only if J is allowed and $|J| = i$.*

Proof. Clearly it is true that $q_J(\alpha) \in [f(q_J(\alpha)), q_J(\alpha)]$. By Theorem 3.3.4 there is an extension in degree $|J|$ if and only if there is no other subset J' such that $q_{J'}(\alpha) \in [f(q_J(\alpha)), q_J(\alpha)]$. So we want to show that this is equivalent to J being allowed. Assume J is allowed and take $J' \neq J$. Because the antichain is *boolean* it suffices to consider $J' \supset J$. Let j be in $J' \setminus J$. Then

$$(q_{J'}(\alpha))_{x_j} = \lambda_j - 1 < \lambda_j = (q_J(\alpha))_{x_j}$$

where x_j is the end of the coefficient λ_j in α. Because J is allowed x_j is also the end of the coefficient λ_j in $q_J(\alpha)$. By condition (ii) of Proposition 5.1.1 the inequality above implies that $q_{J'}(\alpha) \notin [f(q_J(\alpha)), q_J(\alpha)]$. Reciprocally, assume that J is not allowed. Then there exists $j \in S_\alpha$ such that $j \in J$, $j-1 \notin J$ and $\lambda_{j-1} = \lambda_j - 1$. One can then check that $q_{J \sqcup \{j-1\}}(\alpha) \in [f(q_J(\alpha)), q_J(\alpha)]$ using the characterisation for the support of the $\mathcal{P}_\alpha$ and conclude with Proposition 5.1.1. Figure 5.4 provides an illustration for the arguments of the proof. □

Notation 5.1.9. There exists a *canonical realisation* of these extensions defined as morphisms of complexes between the projective resolution associated to α and the interval module associated to $q_J(\alpha)$ concentrated in degree $|J|$. We take the morphisms of modules in degree $|J|$ to be the canonical projection of the factor $P_{q_J(\alpha)}$ onto $[f(q_J(\alpha)), q_J(\alpha)]$ and write the resulting morphism of complexes u^α_J. Among the extensions we just exhibited we are particularly interested in the ones where $J = \{i\} \subseteq S_\alpha$ and write these ${}^1u^\alpha_i$.

Up to homotopy there exists a unique lift of these morphisms along the projective resolution $\mathcal{P}_{q_J(\alpha)}[|J|]$.

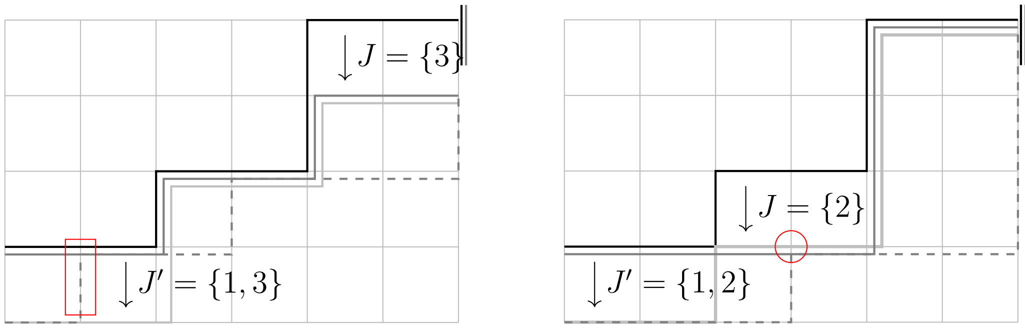


Figure 5.4: Illustration of the proof of Fact 5.1.8

Lemma 5.1.10. *Let α be a right enhanced partition and let J be an allowed subset of S_α of size j. The lift ϕ of $u_J^\alpha : \mathcal{P}_\alpha \to \mathcal{P}_{q_J(\alpha)}[|J|]$ has support $\bigoplus_{J\subseteq I} P_{q_I(\alpha)}$ in each degree. More precisely, in degree $j+k$, we have*

$$\phi_{j+k} = \bigoplus_{\substack{J\subset I\\ |I|=j+k}} (-1)^{|I\setminus J|_J + |J|\cdot k} \cdot id_{P_{q_I(\alpha)}}$$

where $|I|_J = \sum_{i\in I} |i|_J$.

Proof. The morphism of complexes $\phi = (\phi_l)_l$ is concentrated in degrees greater than the cardinal $|J| = j$ of J. In each degree, because the antichains are strong, the morphism of module ϕ_{j+k} decomposes as linear combinations of the identity morphisms of the indecomposable projective summands appearing in both the source and the target. These are precisely the summands $P_{q_I(\alpha)}$ satisfying $J \subseteq I$. See Lemma 5.1.7 for more details. It remains to determine the coefficients $\varepsilon(I)$ associated to each such summand. Assume we have computed the coefficient for degrees between j and $j+k$. We want to compute $\varepsilon(I \sqcup \{i\})$ where i is an element of S_α and I is of cardinal k. First we assume that $\lambda_2 > 1$ or that $2 \notin J$. The relevant morphisms fit in the following commutative square.

$$\begin{tikzcd}
P_{q_{I\sqcup\{i\}}(\alpha)} \arrow[r, "(-1)^{|i|_{I\sqcup\{i\}}}"] \arrow[d, "\varepsilon(I\sqcup\{i\})"'] & P_{q_I(\alpha)} \arrow[d, "\varepsilon(I)"] \\
P_{q_{I\setminus J\sqcup\{i\}}(q_J(\alpha))} \arrow[r, "(-1)^{j+|i|_{I\setminus J\sqcup\{i\}}}"] & P_{q_{I\setminus J}(q_J(\alpha))}
\end{tikzcd} \tag{5.1}$$

Note that the bottom boundary map has a $(-1)^j$ sign because the complex is shifted by

j. For the square to commute it must be that

$$(-1)^{j+|i|_{I\setminus J\sqcup\{i\}}} \times \varepsilon(I \sqcup \{i\}) = (-1)^{|i|_{I\sqcup\{i\}}} \times \varepsilon(I).$$

Recall from equation (1.6) that

$$|i|_I = \{h \in I | h \leq i\} = |i|_{I\setminus J} + |i|_J.$$

Hence we get a recursive formula for the coefficient $\varepsilon(I \sqcup \{i\}) = (-1)^{j+|i|_J} \times \varepsilon(I)$. By setting the degree j sign to be 1, we necessarily get the following formula for the degree $j+k$ maps

$$\phi_{j+k} = \bigoplus_{\substack{J\subset I \\ |I|=j+k}} (-1)^{|I\setminus J|_J+|J|\cdot k} \cdot id_{P_{q_I(\alpha)}}$$

where we introduce the notation $|I|_J = \sum_{i\in I} |i|_J$. The resulting coefficient is the product of the contribution of the previous signs which results in the two sums in the exponent. It is independent of the choice of i made before. It is then clear that this indeed yields a morphism of complexes. Just like in Lemma 5.1.7, when $\lambda_2 = 1$ and $2 \in J$ replace $I \setminus J$ by $I \setminus J - 1$ to have the corresponding subset of $S_{q_J(\alpha)}$. The index 2 is the minimum of J and of I. We can then check that we have

$$|i|_I = \{h \in I | h \leq i\} = |i|_{I\setminus J-1} + |i|_J$$

which yields the same recursive formula and thus the same coefficients for the lift. In both cases there is no non trivial homotopy for the pair of complexes, so the explicit form we provided is in fact the only one and we shall refer to it several times in what follows. To conclude this proof we underline the fact that $\varepsilon(I)$ is indeed not zero for all subset I of S_α containing J.

□

Similarly, it is possible to determine precisely the lift of the degree zero morphisms along the projective resolutions. We will do this at the end of this section.

Lemma 5.1.11. *Let $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ and $\beta = (l_1^{m_1}, \ldots, l_s^{m_s} | n^{m_{s+1}})$ be two partitions and let J be a subset of S_α. The subset J is the unique subset of S_α such that* $q_J(\alpha) \in [f(\beta), \beta]$ if and only if

(a) there exists a morphism in degree zero from $\mathcal{P}_{q_J(\alpha)}$ to $\mathcal{P}_\beta$,

(b) the set J is allowed,

(c) *if* $\epsilon \in J$ *and* $\lambda_\epsilon = 1$ *then the partition* β *should have its first value zero and* $y_1 > x_{\epsilon-1}$.

Proof. Assume J is the unique subset of S_α satisfying $q_J(\alpha) \in [f(\beta), \beta]$. First we will prove that J is allowed by contraposition. Assume there exists $j \in J$ such that $j - 1 \notin J$. If such an element j does not exist then J is automatically allowed. From these two assumptions it follows that $q_{J \sqcup \{j-1\}}(\alpha) \notin [f(\beta), \beta]$ and $q_{J \setminus \{j\}}(\alpha) \notin [f(\beta), \beta]$. Hence

- there exists k such that $l_k = \lambda_j - 1$ and $x_{j-1} < y_k \leq x_j$;
- there exists k' such that $l_{k'} = \lambda_{j-1}$ and $x_{j-2} < y_{k'} \leq x_{j-1}$.

In particular, $k \neq k'$ and $\lambda_{j-1} \neq \lambda_j - 1$. This proves that J is allowed. Next we show that there is a degree zero morphism. Note that the statement above about k is true for any element j of J and the one about k' for any $j' \notin J$. It follows that

$$\{\lambda'_i | i \in S_{q_J(\alpha)}\} \subseteq \{l_j | j \in S_\beta\}.$$

By Proposition 5.1.1 Item (iv), there is a non zero morphism from $\mathcal{P}_{q_J(\alpha)}$ to $\mathcal{P}_\beta$. Lastly we check Condition (c). We want to argue that if $\epsilon \in J$, $\lambda_\epsilon = 1$ and $\beta_{x_{\epsilon-1}} > 0$ then $q_{J \setminus \{\epsilon\}}(\alpha)$ is also in $[f(\beta), \beta]$ as it satisfies all the conditions that define the interval. This would be a contradiction to J being the unique subset of S_α such that $q_{J'}(\alpha) \in [f(\beta), \beta]$. To see this, first notice that because there exists a nonzero morphism from $\mathcal{P}_{q_J(\alpha)}$ to $\mathcal{P}_\beta$, by Proposition 5.1.1, Item (v), the first non zero value of β, is also the first non zero value of $q_J(\alpha)$. Moreover, in β, it runs beyond the index x_ϵ *i.e.* the first non zero value in α. Hence, we would still have $q_{J \setminus \{\epsilon\}}(\alpha) \leq \beta$. Moreover, $q_{J \setminus \{\epsilon\}}(\alpha)$ and $q_J(\alpha)$ have the same values after the index x_ϵ and the defining conditions of the interval $[f(\beta), \beta]$ are checked at indices y_i which all appear after x_ϵ. Hence the first non zero value of β starts before $x_{\epsilon-1} + 1$.

Conversely, assume that J is allowed, satisfies Condition (c) and that there is a morphism from $\mathcal{P}_{q_J(\alpha)} \to \mathcal{P}_\beta$. Consider a subset J' of S_α and suppose that $q_{J'}(\alpha) \in [f(\beta), \beta]$. We will show that $J' = J$. By Proposition 5.1.1 for all non empty $J'' \subseteq S_{q_J(\alpha)}$, the element $q_{J''}(q_J(\alpha))$ is not in $[f(\beta), \beta]$. Because the element

$$q_{J'}(\alpha) \wedge q_J(\alpha) = q_{J \cup J'}(\alpha) = q_{J' \setminus J}(q_J(\alpha))$$

is in the interval $[f(\beta), \beta]$ as well, we conclude that $J' \setminus J = \varnothing$ and $J' \subseteq J$. To see that in fact $J = J'$, say there exists $j \in J \setminus J'$. Take j to be minimal and assume first that $j \neq 2$ or that $\lambda_2 > 1$. Because J is allowed, the partitions $q_J(\alpha)$ and $q_{J'}(\alpha)$ have the same non zero coefficients and the same multiplicities for non zero coefficients up until the index x_{j-1}. Moreover, the j^{th} coefficient of $q_J(\alpha)$, $\lambda_j - 1$, runs from $x_{j-1} + 1$ to x_j and no further. The j^{th} coefficient of $q_{J'}(\alpha)$, λ_j, runs from $x_{j-1}+1$ to x_j. This follows from J being allowed, but we point out that it could extend further as the subset J' needs not be allowed. This is why taking j minimal helps avoid these problems as much as possible. By Proposition 5.1.1 there exists $k \in S_\beta$ such that $x_{j-1} < y_k \leq x_j$ and $l_k = \lambda_j - 1$. If we assume that $q_{J'}(\alpha) \in [f(\beta), \beta]$ then we would also have $l_k = \lambda_j$ which is impossible hence $J' = J$.

If $\lambda_2 = 1$ and $2 \in J$, then our assumption on the zero value in β implies that 2 is necessarily in J'. Hence the previous argument concludes the proof. □

Proposition 5.1.12. *Let $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta[i]$ be a non zero morphism in $\mathcal{Y}_{m,n}$. Then there exists a unique subset J of S_α such that ϕ factors through $\mathcal{P}_{q_J(\alpha)}[i]$ and $|J| = i$ completing the following commutative diagram.*

$$\begin{tikzcd}
\mathcal{P}_\alpha \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{P}_\beta[|J|] \\
& \mathcal{P}_{q_J(\alpha)}[|J|] \arrow[ur] &
\end{tikzcd}$$

Proof. By Theorem 3.3.4, there exists a non zero morphism ϕ if and only if there exists a unique subset J such that $q_J(\alpha) \in [f(\beta), \beta]$. Moreover any two maps between $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta[i]$ differ by multiplication by a scalar. By Lemma 5.1.11 the following morphisms exist and are unique up to multiplication by a scalar

$$\mathcal{P}_\alpha \to \mathcal{P}_{q_J(\alpha)}[|J|] \to \mathcal{P}_\beta[|J|].$$

It remains to check that their composition is non zero in the homotopy category. We represent the morphisms in the following diagram.

$$\begin{tikzcd}
\mathcal{P}_\alpha : \arrow[d] \arrow[dd, dashed, bend right, "?"'] & \cdots \arrow[r] \arrow[d] & \bigoplus\limits_{|I|=|J|} P_{q_I(\alpha)} \arrow[r] \arrow[d] \arrow[dd, dashed, bend right, "?"'] & \bigoplus\limits_{|I|=|J|-1} P_{q_J(\alpha)} \arrow[r] \arrow[d] \arrow[ddl, dashed, "0"'] & \cdots \arrow[r] & P_\alpha \\
\mathcal{P}_{q_J(\alpha)}[|J|] : \arrow[d] & \cdots \arrow[r] \arrow[d] & P_{q_J(\alpha)} \arrow[r] \arrow[d] & 0 \arrow[d] & & \\
\mathcal{P}_\beta : & 0 \arrow[r] & {[f(\beta), \beta]} \arrow[r] & 0 & &
\end{tikzcd}$$

The composition of the two module morphisms in degree $|J|$ is non zero because the support of the top map is $P_{q_J(\alpha)}$ and $q_J(\alpha) \in [f(\beta), \beta]$. No homotopy map can be constructed because by assumption if $I \neq J$, $q_I(\alpha) \notin [f(\beta), \beta]$. By Theorem 3.3.4 the resulting non zero morphism in the homotopy category is proportional to ϕ and the result follows. $\square$

Next we decompose further the extensions $\mathcal{P}_\alpha \to \mathcal{P}_{q_J(\alpha)}[|J|]$ and the degree zero morphisms.

Lemma 5.1.13. *Let α be a partition and J be an allowed subset of S_α. Then the extension $u_J^\alpha : \mathcal{P}_\alpha \to \mathcal{P}_{q_J(\alpha)}[k]$ discussed in Proposition 5.1.8 decomposes as*

$$\mathcal{P}_\alpha \xrightarrow{{}^1u_{j_1}} \mathcal{P}_{q_{\{j_1\}}(\alpha)}[1] \xrightarrow{{}^1u_{j_2}[1]} \dots \xrightarrow{{}^1u_{j_k}[k-1]} \mathcal{P}_{q_J(\alpha)}[k],$$

where $J = \{j_1, \dots j_k\}$ is totally ordered by $j_t < j_{t+1}$.

Proof. First notice that the truncated sets $J_i = \{j_1, \dots, j_i\}$ are all allowed. We proceed by induction on k. Assume the result holds for $k-1$. Then by Theorem 3.3.4 it suffices to show that the composition

$$\mathcal{P}_\alpha \to \mathcal{P}_{q_{J_{k-1}}(\alpha)}[k-1] \to \mathcal{P}_{q_J(\alpha)}[k]$$

is non zero. Again we can draw a diagram to visualise the situation:

$$\begin{tikzcd}
\mathcal{P}_\alpha : \arrow[d] \arrow[dd, dashed, bend right, "?"'] & \cdots \arrow[r] \arrow[d] & \bigoplus\limits_{|I|=k} P_{q_I(\alpha)} \arrow[r] \arrow[d] \arrow[dd, dashed, bend right, "?"'] & \bigoplus\limits_{|I|=k-1} P_{q_I(\alpha)} \arrow[r] \arrow[d] \arrow[ddl, dashed, "0"] & \bigoplus\limits_{|I|=k-2} P_{q_I(\alpha)} \arrow[d] \\
\mathcal{P}_{q_{J_{k-1}}(\alpha)}[k-1] : \arrow[d] & \cdots \arrow[r, dashed] \arrow[d] & \bigoplus\limits_{i\in S_{q_{J_{k-1}}(\alpha)}} P_{q_i(q_{J_{k-1}}(\alpha))} \arrow[d] \arrow[r, dashed] & P_{q_{J_{k-1}}(\alpha)} \arrow[r] \arrow[d] & 0 \\
\mathcal{P}_{q_J(\alpha)}[k] : & 0 \arrow[r] & {[f(q_J(\alpha)), q_J(\alpha)]} \arrow[r] & 0 &
\end{tikzcd}$$

It is clear that $q_I(\alpha) \not\leq q_J(\alpha)$ when $|I| = k-1$. It might not be clear however that the composition of the two module maps in degree k is non zero. Because the bottom map has support $P_{q_J(\alpha)}$ we need to argue that the restriction of the top map to $P_{q_J(\alpha)}$ is non zero either. This is the case by Lemma 5.1.10. Finally, we need to point out that this results in the canonical map u_J and not just a map proportional to it meaning that no negative sign appears when composing the two maps. We compute the sign of the module map in degree j with Lemma 5.1.10. It has support the indecomposable projective module $P_{q_J(\alpha)}$. Its sign is

$$(-1)^{|j_k|_{J_{k-1}}+1\cdot(k-1)} = (-1)^{2\cdot(k-1)} = 1$$

Hence no sign appears. □

A similar result holds for degree zero morphisms.

Lemma 5.1.14. *Let $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta$ be a non zero morphism of modules. Then there exists a sequence $(d_1, \dots, d_r, d_\star)$ such that for all $0 \leq i < r$, we have $0 \leq d_i \leq \mu_i$, with $d_i = \mu_i$ only if $i = 1$ and $\lambda_1 = 0$, and if $\lambda_r = n$, we have $d_\star = 0$, otherwise $d_\star \in \{0, 1\}$ so that*

$$\beta = p_1^{d_1} \circ \cdots \circ p_r^{d_r} \circ p_\star^{d_\star}(\alpha).$$

Moreover, the morphism factors through each of the objects associated to the intermediate partitions.

Proof. By Theorem 3.3.4 and Item (v) of Proposition 5.1.1 the inclusion

$$\{\lambda_i | i \in S_\alpha\} \subseteq \{l_j | j \in S_\beta\}$$

holds. In other words, for all $i \in S_\alpha$, there exists $j \in S_\beta$ such that $\lambda_i = l_j$. Moreover their ending indices satisfy $x_{i-1} < y_j \leq x_i < y_{j+1}$. For $i \in S_\alpha$ we set $d_i = x_i - y_j$. If $l_1 = 0$, because $\alpha \leq \beta$ we also have $\lambda_1 = 0$ and $y_1 \leq x_1$ so set $d_1 = x_1 - y_1$. Lastly, we set $d_\star = 1$ if $\lambda_r < n$, $\mu_{r+1} > 0$ and $m_{s+1} = 0$. Otherwise, $d_\star = 0$. The condition $d_i < \mu_i$ holds because $x_{i-1} < y_j$ when $i \neq 1$. The resulting partition has the same coefficients as β. The multiplicities also match when $i \leq r$:

$$x_i - d_i - x_{i-1} - d_{i-1} = x_i - x_i + y_j - x_{i-1} + x_{i-1} - y_{j-1} = m_j.$$

To see that the map factors through the intermediate partitions use Item (iv) of Proposition 5.1.1: let k be in S_α, $0 \leq d < d_k$ and set $\gamma = p_k^d \circ p_{k+1}^{d_{k+1}} \circ \cdots \circ p_r^{d_r} \circ p_\star^{d_\star}(\alpha)$ and assume the morphism $\mathcal{P}_\alpha \to \mathcal{P}_\gamma$ can be decomposed through all the intermediate links. By construction, $\alpha \in [f(p_k(\gamma)), \gamma] \cap [f(\gamma), \gamma] \cap [f(\alpha), \alpha]$ so the map induced by the composition

$$\mathcal{P}_\alpha \to \mathcal{P}_\gamma \xrightarrow{{}^0u_k^\gamma} \mathcal{P}_{p_k(\gamma)}$$

is non zero on this vertex hence the composition itself is non zero. Recall that by Theorem 3.3.4 the hom spaces are one dimensional. It follows that the map ϕ can be factored into a sequence of ${}^0u_k^{\alpha'}$ maps. □

Morphisms in the category $\mathcal{Y}_{m,n}$ can thus be decomposed into a composition of extensions ${}^1u_j^{\alpha'}$ and morphisms ${}^0u_k^{\alpha'}$. Most of them cannot be decomposed further.

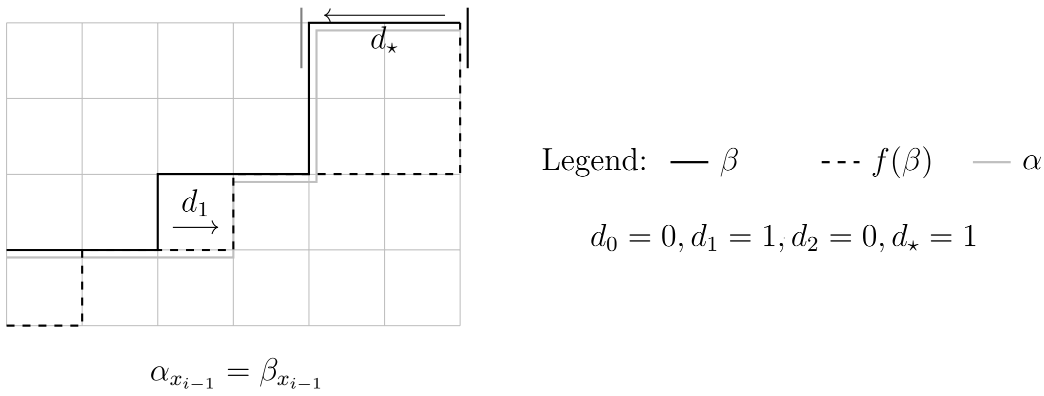


Figure 5.5: Illustration of the proof of Lemma 5.1.14

Lemma 5.1.15. *Let α be a partition, $h \in S_\alpha$ allowed and $k \in \{1, \dots, r\} \cup \{\star\}$ such that p_k is well defined. Assume that $h \neq \epsilon$ or that $\lambda_2 > 1$. Then the morphisms ${}^1u_h$ and ${}^0u_k$ are irreducible.*

Proof. The situation is summed up in the following diagrams.

$$\begin{tikzcd} \mathcal{P}_\alpha \arrow[rr, "{}^1u_h"] \arrow[dr] & & \mathcal{P}_{q_h(\alpha)}[1] \\ & \mathcal{P}_\beta[d] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}_\alpha \arrow[rr, "{}^0u_k"] \arrow[dr] & & \mathcal{P}_{p_k(\alpha)} \\ & \mathcal{P}_\beta \arrow[ur] & \end{tikzcd}$$

Consider a morphism in degree zero of the form ${}^0u_k$ and assume it factors through an object $\mathcal{P}_\beta$ as $f \circ g$. Letters λ_i and x_i refer to the partition α, λ'_i and x'_i to $p_k(\alpha)$ and l_j and y_j to β. Then, by Proposition 5.1.1 Item (v), it holds that

$$\{\lambda'_i | i \in S_{p_k(\alpha)}\} \subseteq \{l_j | j \in S_\beta\} \subseteq \{\lambda_i | i \in S_\alpha\}$$

and for all $j \in S_\beta$ there exists $i \in S_\alpha$ such that $x_{i-1} < y_j \leq x_i < y_{j+1}$ and $l_j = \lambda_i$. The index $i \in S_\alpha$ associated to the value λ_i in α corresponds to an index $i' \in S_{p_k(\alpha)}$ associated to the same value $\lambda'_{i'} = \lambda_i$. If $k = 1$, $\lambda_1 = 0$ and $\mu_1 = 1$, then $i' = i - 1$, otherwise, $i = i'$. Note that it is possible that $i = r + 1$ when $k = \star$. In that case, i' can equally correspond to an enhanced or unenhanced coefficient. In turn, for such an i' there exists j' such that $y_{j'-1} < x'_{i'} \leq y_{j'} < x'_{i'+1}$ and $\lambda'_{i'} = l_{j'}$.

It follows that $j = j'$, and that $x'_{i'} \leq y_j \leq x_i$. Hence, when $i \neq k$ we have $x'_{i'} = y_{j'} = x_i$. Otherwise, when $i = k \neq \star$, y_j can be either x_i or $x'_{i'} = x_i - 1$ which means that $\beta = \alpha$ or $\beta = p_k(\alpha)$ and ${}^0u_k$ is irreducible. When $k = \star$, It is easier to use Item (ii) of Proposition 5.1.1. We then conclude that α, $p_k(\alpha)$ and β have the same underlying plain partition and that $f(\alpha) \leq f(\beta) \leq f(p_k(\alpha))$. The plain partitions $f(\alpha)$ and $f(p_k(\alpha))$ differ only at index m. Knowing already that β has the same underlying plain partition as the other two we deduce that $f(\beta) = f(\alpha)$ or $f(\beta) = f(p_k(\alpha))$. This determines the enhancement of β so that $\beta = \alpha$ or $\beta = p_k(\alpha)$.

Next consider an irreducible extension ${}^1u_h$ with α and β as before. The object $\mathcal{P}_\beta$ is shifted by either 0 or 1. We consider first the case where it is shifted by 0. On the one hand, α and β satisfy Item (v) of Proposition 5.1.1. On the other hand by Theorem 3.3.4 and Proposition 5.1.11 there exists f such that $q_f(\beta)$ and $q_i(\alpha)$ satisfy Item (v) of Proposition 5.1.1. Write l'_j the coefficients of $q_f(\beta)$. Combining these observations, we get that for all $i \in S_\alpha$ there exists j such that $\lambda_i = l_j$ and $x_{i-1} < y_j \leq x_i < y_{j+1}$. For such a j there exists i' such that $l'_j = \lambda'_{i'}$ and $y_{j-1} < x_{i'} \leq y_j < x_{i'+1}$. Whether $i = h$ or not, we get that $i = i'$, $y_j = x_i$ and $\beta = \alpha$. The case where $\mathcal{P}_\beta$ is shifted by one is very similar: β and $q_h(\alpha)$ satisfy Item (v) of Proposition 5.1.1 and by Proposition 5.1.11 there exists a unique $f \in S_\alpha$ allowed such that $q_f(\alpha)$ and β satisfy Item (v) of Proposition 5.1.1. Write λ''_i the coefficients of $q_f(\alpha)$. Combining these observations, we get that for all $i \in S_\alpha$ there exists j such that $\lambda''_i = l_j$ and $x_{i-1} < y_j \leq x_i < y_{j+1}$. For such a j there exists i' such that $l_j = \lambda'_{i'}$ and $y_{j-1} < x_{i'} \leq y_j < x_{i'+1}$. Whether $i = h$ or not, we get that $i = i'$, $y_j = x_i$ and $\beta = q_h(\alpha)$. $\square$

Notes 5.1.16. If $\lambda_2 = 1$, then ${}^1u_2$ decomposes into ${}^1u_1 \circ {}^0u_1^{\mu_1 - 1} \circ {}^0u_0$ where μ_1 is the multiplicity of the value zero in the source partition. We give a concrete example below.

$$\begin{tikzcd}
\mathcal{P}_{(0011)} \arrow[rr] \arrow[dr] & & \mathcal{P}_{(0000)}[1] \\
& \mathcal{P}_{(1111)}[1] \arrow[ur] &
\end{tikzcd}$$

We invite the reader to look at Figure 5.1 again as each commutative triangle illustrates one of Proposition 5.1.12, Lemma 5.1.13, or Lemma 5.1.14. The figure also corroborate 5.1.15 and Notes 5.1.16. Before we move on to describe relations between morphisms, we give an explicit lift of the canonical morphism ${}^0u_i$.

Lemma 5.1.17. *Let $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta$ be a non zero morphism. Then its lift along the projective resolutions is made of monomorphism in each degree and all signs are positive.*

Proof. According to Lemma 5.1.14 we can write β as $p_I(\alpha)$ where $I = (i_1, \ldots, i_k)$ is sequence of elements of $\{1, \ldots, r\} \cup \{\star\}$. Let J be a subset of S_α of size k. If 1 does not appear in I or $\mu_1 > 1$ then J can also be seen as a subset of S_β. Applying Item (v) of Lemma 5.1.7 repeatedly, the subset J is then the only subset of S_β satisfying

$q_J(\alpha) \leq q_J(\beta)$. If $1 \in J$ and $\mu_1 = 1$, we apply Item (v) and (vi) of Lemma 5.1.7 to get that $J - 1$ is the only subset of size k of S_β for which $q_J(\alpha) \leq q_{J-1}(\beta)$. Hence, when 1 does not appear in I or $\mu_1 > 1$ we have

$$\phi_k = \oplus_{J \subseteq S_\alpha} c_J \iota_{q_J(\alpha)}^{q_J(\beta)}$$

otherwise it is

$$\phi_k = \oplus_{J \subseteq S_\alpha} c_J \iota_{q_{J-1}(\alpha)}^{q_J(\beta)}$$

where $c_J \in \Bbbk$. It remains to see that $c_J = 1$ makes all the squares commute similarly to the proof of Lemma 5.1.10. This time the boundary maps are identical in the top complex and the bottom complex hence no signs appear. Hence we have an explicit lift of the canonical degree zero map which is a monomorphism in each degree. □

5.2 Configurations and relations

We can now describe the irreducible morphisms of Section 5.1 using configurations through the map ϕ_r from Section 4.1. To do so we define a partial function on configurations, as in [31] for the construction of Higher Auslander algebras of type A. If R is a configuration, $k \in R$ and $k - 1 \notin R$, we write

$$\sigma_k^-(R) = (R \cup \{k-1\}) \setminus \{k\}. \tag{5.2}$$

Note that if $k = -m$, $\sigma_k^-(R) = (R \cup \{n\}) \setminus \{-m\}$ as k represents an element of $\mathbb{Z}/(m+n+1)\mathbb{Z}$. From now on, we will also write the objects of the category $\mathcal{Y}_{m,n}$ as $\mathcal{P}_R$, implicitly identifying α with R.

Example 5.2.1. Recall the right abacus associated to the partition $(0, 2, 3, 7, 7|)$ from Example 4.1.3.

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
	•				•		•	•				•

Applying σ_{-4}^- to the configuration associated to a consists of taking the bead placed in -4 and sliding it down to the -5. We then get the following abacus,

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
•					•		•	•				•

which is associated to the partition $(0, 2, 3, 7| 7)$.

Proposition 5.2.2. *In terms of configurations, the irreducible morphisms, in degree zero are arrows*

$$\mathcal{P}_R \to \mathcal{P}_{\sigma_k^-(R)} \text{ with } 0 \geq k \in R.$$

Next the irreducible morphisms in degree 1 correspond to the following transformation on configurations

$$\mathcal{P}_R \to \mathcal{P}_{\sigma_k^-(R)} \text{ with } 0 < k \in R.$$

Note that, for a partition α with $\lambda_2 = 1$, the extension ${}^1u_2^\alpha$ does not appear in the statement as the transformation q_2 of the partition does not correspond to a well defined transformation σ_1^- of the corresponding configuration. This is coherent with Note 5.1.16.

Proof. First we prove the statement about the degree zero irreducible morphisms. Let α be a partition. Recall from Corollary 5.1.3 and Lemma 5.1.14 that these morphisms were of the form

$$\mathcal{P}_\alpha \to \mathcal{P}_{p_i(\alpha)}$$

with $i \in \{1, \ldots, r\} \cup \{\star\}$ and $\mu_i > 1$ when $i \neq 1$. First, taking $i \neq \star$ and using 5.1.3, we compute

$$x_j^{p_i(\alpha)} = \begin{cases} x_i - 1 & \text{if } j = i \\ x_j & \text{otherwise.} \end{cases} \tag{5.3}$$

This implies that $R_{p_i(\alpha)} = (R_\alpha \cup \{-x_i\}) \setminus \{-x_i + 1\} = \sigma^-_{-x_i+1}(R_\alpha)$. Now take $i = \star$, with $\lambda_r < n$ and $\mu_{r+1>0}$. The degree zero morphisms corresponding to the transformation $p_\star$ moves the enhancement bar all the way to the right. On the configuration this amounts to removing the coefficient $-m$ and adding n. Finally the degree 1 morphisms were

$$\mathcal{P}_\alpha \to \mathcal{P}_{q_i(\alpha)}[1]$$

where $\lambda_i - 1 > \lambda_{i-1}$. Here the configuration associated $q_i(\alpha)$ is clearly the configuration associated to α where the λ_i was replaced by $\lambda_i - 1$.

Reciprocally, every $k \in R$ such that $k - 1 \notin R$, $\sigma_k^-(R)$ is well defined and corresponds to either an irreducible morphism or an irreducible extension through the inverse of ϕ_r (recall Figure 4.1). If $k \leq 0$, then $\sigma_k^-(R)$ corresponds to the transformation $p_i(\alpha)$ if $k-1$ is the i^{th} gap in the negative side counting from the right. It is associated to ${}^0u_i$. If $k > 0$, then $\sigma_k^-(R)$ corresponds to the transformation $q_j(\alpha)$ if k is the j^{th} positive element of the configuration, in increasing order hence it is associated to ${}^1u_j$. This concludes the proof. □

Note that configurations make the description of the irreducible morphisms more homogeneous. This will also be the case for the relations between them. To simplify notation, the morphism (in degree zero or one) from $\mathcal{P}_R$ to $\mathcal{P}_{\sigma_k^-(R)}$ will be denoted u_k^R since the degree of the morphism is encoded in the sign of k. We will now express the relations between the morphisms and extensions in the language of configurations.

Lemma 5.2.3. *Let R be a configuration, take $k, l \in R$ such that $k-1, l-1 \notin R$. We have the following equalities in the category $\mathcal{Y}_{m,n}$:*

$$\rho_{k,l}^R : u_l^{\sigma_k^-(R)} \circ u_k^R = \varepsilon u_k^{\sigma_l^-(R)} \circ u_l^R \text{ and } z_k^R : u_{k-1}^{\sigma_k^-(R)} \circ u_k^R = 0 \tag{5.4}$$

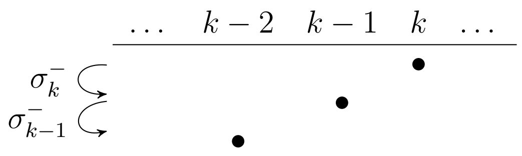
where $\varepsilon = -1$ if k and l are positive and 1 otherwise.

Proof. We distinguish several cases depending on the sign of the integers k and l. First consider $\rho_{k,l}^R$ when both k and l are positive *i.e.* the irreducible morphisms are concentrated in degree 1. Then the morphisms in question can be summed up in the following diagram. We use partitions because the order relation is not clear on configurations. The elements k and l of R can be uniquely associated to $i, j \in S_\alpha$

$$\begin{tikzcd}
\mathcal{P}_\alpha : \arrow[d] \arrow[dd, bend right, dashed, "?"'] & \cdots \arrow[r] \arrow[d] & \bigoplus_{i',j'\in S_\alpha} P_{q_{\{i',j'\}}(\alpha)} \arrow[r] \arrow[d] \arrow[dd, bend left, dashed, "?"] & \bigoplus_{i'\in S_\alpha} P_{q_{i'}(\alpha)} \arrow[r] \arrow[d] & P_\alpha \arrow[r] \arrow[d] & 0 \\
\mathcal{P}_{q_i(\alpha)} : \arrow[d] & \cdots \arrow[r] \arrow[d] & \bigoplus_{j'\in S_\alpha} P_{q_{j'}(q_i(\alpha))} \arrow[r] \arrow[d] & P_{q_i(\alpha)} \arrow[r] \arrow[d] & 0 & \\
\mathcal{P}_{q_{\{i,j\}}(\alpha)} : & 0 \arrow[r] & \left[f(q_{\{i,j\}}(\alpha)), q_{\{i,j\}}(\alpha)\right] \arrow[r] & 0 & &
\end{tikzcd} \tag{5.5}$$

The resulting morphism of modules in degree two is the morphism from $\mathcal{P}_\alpha$ to $\mathcal{P}_{q_{\{i,j\}}(\alpha)}$ described in 5.1.8 up to a sign because i and j are assumed to be allowed. This is symmetric in i and j hence the result. To be more specific, according to Lemma 5.1.10, signs appears in the upper middle square. The boundary map component going from $P_{q_{\{i,j\}}(\alpha)}$ to $P_{q_j(q_i(\alpha))}$ has sign $(-1)^{|j|_{\{i,j\}}}$. Hence the two compositions differ by a factor -1.

Next we consider z_k^R when $k > 2$: the morphisms in question can be summed up by the following diagram using partitions.



$$
\begin{array}{cccc}
\dots & x_i & x_i+1 & x_i+2 & \dots \\
\hline
 & \lambda_i & \lambda_i & \boxed{\lambda_{i+1}} & \lambda_{i+2} \\
 & \lambda_i & \lambda_{i+1} & \lambda_{i+1} & \lambda_{i+2} \\
 & \lambda_i & \lambda_{i+1} & \boxed{\lambda_{i+2}} & \lambda_{i+2}
\end{array}
\begin{array}{l}
\circlearrowright p_i^{-1} \\
\circlearrowright p_{i+1}^{-1}
\end{array}
$$

Figure 5.6: relations between irreducible degree zero morphisms

$$
\begin{tikzcd}
\mathcal{P}_\alpha : \arrow[d] \arrow[dd, dashed, bend right, "?"'] & \dots \arrow[r] \arrow[d] & \bigoplus\limits_{i',j\in S_\alpha} P_{q_{\{i',j\}}(\alpha)} \arrow[r] \arrow[d] \arrow[dd, dashed, bend left, "?"] & \bigoplus\limits_{i'\in S_\alpha} P_{q_{i'}(\alpha)} \arrow[r] \arrow[d] & P_\alpha \arrow[r] \arrow[d] & 0 \\
\mathcal{P}_{q_i(\alpha)}[1] : \arrow[d] & \dots \arrow[r] \arrow[d] & \bigoplus\limits_{j\in S_{q_i(\alpha)}} P_{q_j(q_i\alpha)} \arrow[r] \arrow[d] & P_{q_i(\alpha)} \arrow[r] \arrow[d] & 0 & \\
\mathcal{P}_{q_i(q_i(\alpha))}[2] : & 0 \arrow[r] & {[f(q_i(q_i(\alpha))), q_i(q_i(\alpha))]} \arrow[r] & 0 & &
\end{tikzcd}
\tag{5.6}
$$

There cannot be a morphism of modules in degree 2 because the $q_{\{i',j\}}(\alpha)$ cannot be compared with $q_i(q_i\alpha)$ when $j \neq i$.

We now take k and l non positive. For the $\rho^R_{k,l}$, notice that both branches are non zero when $l \neq k-1$ and conclude by saying that hom spaces are one dimensional.

To see that z_k is a relation recall from 5.1.1 that there exists a morphism between $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ if and only if α and β have the same coefficients up to bars and zeros and $\alpha_{y_j} = \beta_{y_j}$ for all i in S_β. Let α be a partition, such that its corresponding right configuration R contains k but neither $k-1$ nor $k-2$. Say the negative coefficient k is associated to the i^{th} value of the partition. Figure 5.6 illustrates the action of σ^-_k and σ^-_{k-1} on R when $k > -m-1$. On the right side, we see that the coefficients at $y_{i+1} = x_i + 2$ are different in α and β meaning no arrow exists between the first and last partition. When $k = -m-1$ the argument can be summed up with a similar table which we leave to the reader. Finally, when $k = -m$, z_k is indeed a relation because there is no extension from α to the partition $q_{r+1}(p_\star(\alpha))$. The partitions α and $p_\star(\alpha)$ have the same underlying plain partitions but not the same associated antichains and $q_{r+1}(p_\star(\alpha))$ does not appear in the antichain of α.

Next we assume k is positive and l is non positive. We can check that both sides of the square yield the same result by looking at diagrams like 5.5 and 5.6 and discussing the support of the morphisms in degree 1.

$$\begin{tikzcd}
\mathcal{P}_\alpha : \arrow[d] \arrow[dd, bend right, dashed, "\neq 0"'] & \dots \arrow[r] \arrow[d] & \bigoplus_{i'\in S_\alpha} P_{q_{i'}(\alpha)} \arrow[r] \arrow[d] \arrow[dd, bend right, dashed, "\text{supported on } P_{q_i(\alpha)}"'] & P_\alpha \arrow[r] \arrow[d] \arrow[ddl, dashed, "0"] & 0 \\
\mathcal{P}_{q_i(\alpha)}[1] : \arrow[d] & \dots \arrow[r] \arrow[d] & P_{q_i(\alpha)} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
\mathcal{P}_{p_j(q_i(\alpha))}[1] : & 0 \arrow[r] & {[f(p_j(q_i(\alpha))), p_j(q_i(\alpha))]} \arrow[r] & 0 &
\end{tikzcd}$$

Figure 5.7: $\rho_{k,l}$ when $l \leq 0 < k$ (1)

$$\begin{tikzcd}
\mathcal{P}_\alpha : \arrow[d] \arrow[dd, bend right, dashed, "\neq 0"'] & \dots \arrow[r] \arrow[d] & \bigoplus_{i'\in S_\alpha} P_{q_{i'}(\alpha)} \arrow[r] \arrow[d] \arrow[dd, bend right, dashed, "\text{supported on } P_{q_i(\alpha)}"'] & P_\alpha \arrow[r] \arrow[d] \arrow[ddl, dashed, "0"] & 0 \\
\mathcal{P}_{p_j(\alpha)} : \arrow[d] & \dots \arrow[r] \arrow[d] & \bigoplus_{i'\in S_\alpha} P_{q_{i'}(p_j(\alpha))} \arrow[r] \arrow[d] & P_{p_j(\alpha)} \arrow[r] \arrow[d] & 0 \\
\mathcal{P}_{p_j(q_i(\alpha))}[1] : & 0 \arrow[r] & {[f(p_j(q_i(\alpha))), p_j(q_i(\alpha))]} \arrow[r] & 0 &
\end{tikzcd}$$

Figure 5.8: $\rho_{k,l}$ when $l \leq 0 < k$ (2)

$\square$

Example 5.2.4. We give one instance of z_k making a composition of morphisms equal to zero. Take for instance $\alpha = (0, 0, 2, 2, 2, 5, 6, 6, 6, 6|)$. It is associated to the configuration

$$C = \{-9, -8, -7, -4, -3, -1, 0, 2, 5, 6\}.$$

We compute $\sigma^-_{-4}(C) = \{-9, -8, -7, -5, -3, -1, 0, 2, 5, 6\}$ which is associated to the partition $\beta = (0, 0, 2, 2, 5, 5, 6, 6, 6, 6|)$ It is easy to check that there is a morphism between their corresponding modules using Proposition 5.1.1 Item (v). Similarly there is a morphism from the module associated to β to $\sigma^-_{-5}(\sigma^-_{-4}(C)) = \{-9, -8, -7, -6, -3, -1, 0, 2, 5, 6\}$. Its corresponding partition is $\gamma = (0, 0, 2, 2, 5, 6, 6, 6, 6, 6|)$. Then one can check that $a_{x_3^\gamma} \neq c_{x_3^\gamma}$ meaning there is no morphism from $\mathcal{P}_\alpha$ to $\mathcal{P}_\gamma$ and z^C_{-4} is a relation. This illustrates the general argument.

Remark 5.2.5. The fact that some squares commute and other square anticommute is inconvenient for Theorem E. However, in Corollary 5.2.14, we will see that these signs can be corrected.

Definition 5.2.6. A subset J of a configuration R is allowed if for all $j \in J$ either $j-1 \notin R$ or $j-1 \in J$. Then $\sigma_J^-(R)$ is well defined. Note that if J is in the positive side of R, J corresponds to an allowed subset of S_α.

Lemma 5.2.7. *Let $R_0, \dots, R_t$ be configurations such that for all $q \in \{0, \dots, t-1\}$, there exists $k_q \in R_q$ such that $R_{q+1} = \sigma_{k_q}^-(R_q)$. Then the morphism $f = u_{k_{p-1}}^{R_{p-1}} \circ \dots \circ u_{k_0}^{R_0}$ is non zero* only if *for all q, $k_q \in R_0$.*

Proof. Assume that there exists q such that $k_q \notin R_0$. Then there exists $r < q$ with $k_r = k_q + 1$. Take q to be minimal and r as close as possible to q. Take $s \in \{r, \dots, q\}$. Then, by the assumptions on q and r, we either have $k_s > k_q + 1$ or $k_s < k_q$. Whenever s is such that $k_{s+1} < k_q < k_q + 1 < k_s$, equation (5.4) provides the equality

$$u_{k_{s+1}} \circ u_{k_s} = \epsilon u_{k_s} \circ u_{k_{s+1}}.$$

Hence using equation (5.4) we can rewrite the morphism f as

$$f = \epsilon \times (\dots \circ u_{k_q} \circ u_{k_{s(q-1)}} \circ \dots \circ u_{k_{s(\tau+1)}} \circ u_{k_{s(\tau)}} \circ \dots \circ u_{k_{s(r+1)}} \circ u_{k_r} \circ \dots)$$

where s is a permutation on the set $\{r+1, \dots, q-1\}$ and if $i > \tau$, $k_{s(i)} > k_q + 1$ and $k_{s(i)} < k_q$ otherwise. Finally, we use equation (5.4) again to get

$$f = \epsilon \times (\dots \circ u_{k_{s(q-1)}} \circ \dots \circ u_{k_{s(\tau+1)}} \circ u_{k_q} \circ u_{k_r} \circ u_{k_{s(\tau)}} \circ \dots \circ u_{k_{s(r+1)}} \circ \dots)$$

The two central terms correspond to z_{k_r} which is zero by Lemma 5.2.3. This concludes the proof. □

We will show that the converse of this Lemma 5.2.7 is true. To do so we need to find a canonical way of decomposing morphisms into irreducible ones. Knowing that irreducible morphisms are indexed by certain elements of the partition R, we want to equip R with a convenient order relation. Recall that when we defined configurations on $\mathcal{Z}$ we wrote them as increasing sequences using the order relation on $\mathbb{Z}$. This ordering is not adapted to the study of the morphisms: for any partition R containing $-m$ but not n, it is in general unclear whether the morphism u_{-m}^R should come before or after u_k^R when k is positive. Let R and R' be configurations. Suppose that $R' = \sigma_{k_t} \circ \dots \circ \sigma_{k_1}(R)$, that for each $q \leq t$, $k_q \in R$ and the partition $R_q = \sigma_{k_q} \circ \dots \circ \sigma_{k_1}(\alpha)$ is well defined. We have the following composition of irreducible morphisms

$$f = u_{k_t}^{R_{t-1}} \circ \dots \circ u_{k_1}^{R}$$

in $\mathcal{Y}_{m,n}$. Order the combinatorial data $K = (k_1, \ldots, k_p)$ and by the same process the elements of $\mathcal{Z}$ as follows: let k_{min} be the maximal element of $\mathcal{Z}$ for the naive order which does not appear in K and is less than or equal to 1. We consider the total order

$$k_{min} <_f k_{min} + 1 <_f \cdots <_f n <_f -m <_f \cdots <_f k_{min} - 1.$$

Because we picked the combinatorial data so that the consecutive morphisms are well defined, if $k_{min} \neq 1$ then k_{min} is not an element of R. Alternatively, if $k_{min} = 1$, either $1 \in R$ and the first value of the list K ordered with $<_f$ is $k >_f 2$ or $1 \notin R$ and $k >_f 1$. Hence, if k is the minimum of K with regard to $<_f$, then $k - 1$ is not in R.

Lemma 5.2.8. *Consider partitions R, R' and a morphism f as above with combinatorial data $k_1, \ldots, k_t$ and order $<_f$. Suppose also that for all $1 \leq q \leq t$, we have k_q is in R. Then there exists a permutation $p \in \mathfrak{S}_t$ such that we have*

$$k_{p(1)} <_f \cdots <_f k_{p(t)}$$

and

$$f = \varepsilon \cdot u_{k_{p(t)}}^{R_{p(t-1)}} \circ \cdots \circ u_{k_{p(1)}}^{R}$$

for $\varepsilon \in \{1, -1\}$

Proof. Recall from equation (5.4) of Lemma 5.2.3, that for each q, if $k_q, k_{q+1} \in R_{q-1}$ while $k_q - 1 \notin R_{q-1}$ and $k_{q+1} - 1 \notin R_{q-1}$ then

$$u_{k_{q+1}}^{R_q} \circ u_{k_q}^{R_{q-1}} = \epsilon u_{k_q}^{\sigma_{k_{q+1}}(R_q)} \circ u_{k_{q+1}}^{R_{q-1}}$$

with the transformation $\sigma_{k_{q+1}}$ being well defined on R_{q-1}. We want to show that this equation applies when $k_{q+1} <_f k_q$. The fact that $k_q - 1 \notin R_{q-1}$ follows from the assumption that σ_{k_q} is well defined on R_{q-1}. Because $\sigma_{k_{q+1}}$ is well defined on R_q and $k_{q+1} <_f k_q$ it must also be that $k_{q+1} - 1 \notin R_{q-1}$. Using the bubble sort algorithm [15], the chain can be ordered up to a sign by applying relation ρ_{k_{q+1},k_q} when $k_q <_f k_{q-1}$. Indeed the bubble sort algorithm only swaps pairs of entries when they are not ordered according to the relation $<_f$. Moreover the assumptions of the lemma are maintained at each swap. □

Example 5.2.9. We take $R = \{-8, -4, -1, 0, 1, 3, 4, 5\}$ and $K = (-4, -1, 0, 1)$. Then $k_{min} = -2$. The canonical ordering for a chain of morphism defined by data K, is $(-1, 0, 1, -4)$.

We can now tackle the converse of Lemma 5.2.7

Lemma 5.2.10. *Let $R_0, \dots, R_t$ be configurations such that for all $q \in \{0, \dots, t-1\}$, there exists $k_q \in R_q$ such that $R_{q+1} = \sigma^-_{k_q}(R_q)$. Then the morphism $f = u^{R_{p-1}}_{k_{p-1}} \circ \cdots \circ u^{R_0}_{k_0}$ is non zero if and only if for all q, $k_q \in R_0$.*

Proof. Assume that $k_q \in R = R_0$ for all $0 \leq q \leq t-1$. By Lemma 5.2.8 we can assume the elements of the chain are ordered with the ordering $<_f$. Let τ be the maximal index such that $k_\tau \leq_f n$. First we show that $f_{ext} = u_{k_\tau} \circ \cdots \circ u_{k_1}$ is an allowed and non zero extension. Because $k_{min} \notin R$ or $k_1 > 1$ (see remark before Lemma 5.2.8), the set $\{k_1, \dots, k_\tau\}$ is allowed as per Definition 5.2.6 and see section 4.1. So f_{ext} is non zero in $\mathcal{Y}_{m,n}$ by Proposition 5.1.8 and Lemma 5.1.13.

We now consider the morphism $f_0 = u_{k_{t-1}} \circ \cdots \circ u_{k_{\tau+1}}$ which is concentrated in degree zero. To see that it is non zero we argue that *its source and its target satisfy the condition of Proposition 5.1.1 Item (v)*. The source of the morphism is associated to configuration $R^\tau = \sigma^-_{k_\tau} \circ \cdots \circ \sigma^-_{k_1}(R)$ and its target to $R' = \sigma^-_{k_{t-1}} \circ \cdots \circ \sigma^-_{k_{\tau+1}}(R_\tau)$. Denote by α and α' the partitions associated to R and R'. We chose the integer τ such that the two configurations are identical in columns k_{min} to $n-1$. It follows that, in terms of partitions we have $\{\lambda_i | i \in S_{\alpha_\tau}\} \subseteq \{l_j | j \in S_{\alpha'}\}$ completing the second part of Proposition 5.1.1 Item (v).

For the first part, let j be an element of $S_{\alpha'}$ its associated value l_j being an element of R' and its ending index y_j corresponding to a the absence of $-y_j$ in R'. Because R' is obtained from R_τ by moving beads of the abacus by one slot to the left, the gap in $-y_j$ was obtained by filling some gap $k \leq -y_j$ in R_τ. The gap k corresponds to an ending index $-x_i$ of the coefficient λ_i in the partition α_τ. Moreover, $-y_i$ must be strictly less than the next gap to the right of k in R_τ because it would require moving a bead from $-x_{i-1}$ to $-x_{i-1} - 1$ and that $-x_{i-1} \notin R_\tau$.

We need to show that $l_j = \lambda_i$. Because beads can be moved twice without contradicting the assumption that the $k_i's$ are in R there are the same number of gaps in R' and R_τ between column $-x_i$ and k_{min}. Moreover, recall that the t^{th} gap in the negative side must correspond to the t^{th} coefficient of the partition. There are three cases depending on whether zero is a value of the source configuration or the second. If both partitions contain the value zero, then all the gaps are in the negative side and $-x_i$ and $-y_j$ correspond to the same bead. If both partitions do not contain zero as well. Lastly it is possible that R_τ contains 0 but R' does not. In that case, the i^{th} coefficient in α_τ is the $(i-1)^{th}$ coefficient of α' and we again have $\lambda_i = l_j$. Hence there exists a non zero morphism from $\mathcal{P}_{R_\tau}$ to $\mathcal{P}_{R'}$. In turn this means that the map f_0 is non zero because of how composition of morphisms between intervals work.

The final step is to argue that $f = f_0 \circ f_{ext}$ is non zero in $\mathcal{Y}_{m,n}$. We have showed that there exists $J \subseteq S_\alpha$ allowed such that there exists a non zero morphism from $\mathcal{P}_{q_J(\alpha)}$ to $\mathcal{P}_{\alpha'}$. Moreover, if $1 = \lambda_\epsilon \in R$ and $\epsilon \in J$, then 0 is an element of R' because the bead associated to 1 can only be moved once. The same is true for all the beads associated to zero if it was an element of R. Hence if $\epsilon \in J$ and $\lambda_\epsilon = 1$, then the partition β should have its first value be zero *i.e.* $l_1 = 0$ and $y_1 > x_{\epsilon-1}$ and we can apply Lemma 5.1.11 to conclude that there exists a morphism in the derived category from $\mathcal{P}_\alpha$ to $\mathcal{P}_{\alpha'}$. Moreover, by Theorem 3.3.4, the set J that we have identified is the unique subset of S_α such that $q_J(\alpha) \in [f(\alpha'), \alpha']$. By the proof of Proposition 5.1.12, the composition $f = f_0 \circ f_{ext}$ is non zero. This concludes the proof.

□

Remark 5.2.11. By Lemmas 5.1.11, 5.1.13 and 5.1.14 all morphisms decompose into an extension followed by a degree zero morphism. In turn these can be further decomposed into the irreducible morphisms we have identified. The list of the combinatorial data of the morphisms we obtain is called the canonical list associated to a non zero morphism. Its ordering coincides with $<_f$. Its positive (in the naive sense) elements are the elements of the unique subset J of S_α such that $q_J(\alpha) \in [f(\beta), \beta]$.The interlacing of the ending vertices described in Item (v) of Proposition 5.1.1 gives the rest of the canonical list by looking at the transformation associated to the integers in the intervals $\rrbracket -x_i, -y_j \rrbracket$.

As a consequence we have the following proposition

Proposition 5.2.12. *The relations described in equation (5.4) generate the relations between morphisms in the category $\mathcal{Y}_{m,n}$. Hence $\mathcal{Y}_{m,n}$ is generated by quadratic relations.*

Proof. Consider the $\Bbbk$-linear category C defined as follows:

- the objects of C are pairs (R, l) where R is a configuration and l is an integer;
- the morphisms are generated by arrows $(R, l) \to (\sigma_k^-(R), l')$ with $l' = l + 1$ if $k > 0$ and $l' = l$ otherwise,
- with relations $\rho_{k,l}$ and z_k identified in Lemma 5.2.3.

By Lemma 5.2.3 again there is a well defined functor $F : C \to \mathcal{Y}_{m,n}$. This functor is essentially surjective. To prove the current proposition we need to argue that it is an equivalence of categories. By Proposition 5.1.12, Lemma 5.1.14 and Lemma 5.1.13 combined with Proposition 5.2.2 the functor induces surjective maps between the hom spaces $\mathrm{Hom}_C((R,l),(R',l'))$ and $\mathrm{Hom}_{D^b(J_{m,n})}(\mathcal{P}_R,\mathcal{P}_{R'}[l'-l])$. It remains to see that this map is injective. Consider an element

$$\sum_{i=1}^{q} a_i \cdot f_i \text{ of } \mathrm{Hom}_C(R,R') \tag{5.7}$$

where q is an integer, $a_1,\dots,a_q$ are elements of the field $\Bbbk$ and the morphism f_i is a non zero composition of h_k^R morphisms. In other words, there exists a sequence $k_1^i,\dots,k_{p_i}^i$ such that $f_i = h_{k_{p_i}^i} \circ \cdots \circ h_{k_1^i}$. Suppose $F(\sum_{i=1}^q a_i \cdot f_i) = 0$ in $\mathcal{Y}_{m,n}$. We can also assume $f_1 \neq 0$ and $\alpha_1 \neq 0$. The previous Lemma ensures that zero relations, *i.e.* when $q = 1$ correspond exactly to those described in Lemma 5.2.3. To conclude when $q > 1$ we want to argue that sequences $k_1^i,\dots,k_{p_i}^i$ contain the same elements, independently of i.

By Lemmas 5.1.11, 5.1.13 and 5.1.14 there exists a canonical list K of transformations to go from α to β. We argue that it is the only possible list. Suppose L is a list of transformations giving a non zero morphism f from α to β. Order both K and L using the order relation $<_{\min}$ with starting point k_{min} the minimum of the starting points for K and L. If there exists s an element of L which is not in K pick s minimal for $<_{\min}$. Then using the transformations listed by L, $s-1 \in \beta$ but through K, $s-1 \notin \beta$ which is a contradiction. Hence all the elements of L appear in K. Symmetrically, all the elements of K appear in L.

Denote $f^\star$ the composition of the maps associated to the canonical list in increasing order of their combinatorial data. Note that Lemma 5.2.8 and its proof apply to the category C as well since it only uses equation (5.4). Hence every morphism in C from (R,l) to (R',l') is proportional to $f^\star$. In particular, there exist elements $a,b \in \Bbbk$ such that $f_1 - a\cdot f^\star = 0$ in $\mathcal{C}$ and $\frac{1}{a_1}\sum_{i=2} a_i \cdot f_i - b \cdot f^\star = 0$ in $\mathcal{C}$. Because $F(\sum_{i=1}^q a_i \cdot f_i) = 0$ in $\mathcal{Y}_{m,n}$, $a = -b$ and $\sum_{i=1}^q a_i \cdot f_i = 0$ in C. □

Notation 5.2.13. For a configuration R as well as an integer $0 \le l \in R$ and $0 > k \in R$, define

$$\kappa(R,l) = \sum_{\substack{x\in R\\ l\ge x\ge 0}} x \text{ and } \nu(R,k) = \sum_{k \ge x \in R} x.$$

We establish a number of identities concerning κ and ν combined with a transformation σ_l^-. Let l_1 and l_2 be allowed in R_+. Without loss of generality we can assume that $l_1 < l_2$. Then $\kappa(\sigma_{l_1}^-(R), l_2) = \kappa(R,l_2) - 1$ while $\kappa(\sigma_{l_2}^-(R), l_1) = \kappa(R,l_1)$. Hence κ is a combinatorial transformation that detects the event $l_1 \le l_2$. Similarly, if k_1 and k_2 are allowed in R_- and $k_1 < k_2$ then $\nu(\sigma_{k_1}^-(R), k_2) = \nu(R,k_2) - 1$ while $\nu(\sigma_{k_2}^-(R), k_1) = \nu(R,k_1)$. We also have $\kappa(\sigma_k^-(R), l) = \kappa(R,l)$ and $\nu(\sigma_l^-(R), k) = \nu(R,k)$. Alternatively we can define these quantities on partitions. We will only use the quantity κ which we express as follows: let α be a partition, let i be allowed in S_α and set

$$\kappa(\alpha, i) = \sum_{k=1}^{i} \lambda_k. \tag{5.8}$$

Because it will be used several times, we write $\kappa_\alpha = \kappa_R = \kappa(\alpha, r)$.

To conclude this section, we give three presentations of the category $\mathcal{Y}_{m,n}$ by generators and relations. The first one is a direct corollary of Proposition 5.2.12 with the relations of Lemma 5.2.3. According to equation (5.4), with that presentations some square of irreducible morphisms commute and some anticommute. Using Notations 5.2.13 we give two more presentations, one where all the squares commute and one where they all anti-commute.

Corollary 5.2.14. *The morphisms in $\mathcal{Y}_{m,n}$ are generated by*

(i) the maps u_k^R for all $R \in C_{m,n}$ and for all $k \in R$ allowed, with relations generated by $\rho_{k,l}^R = u_l^{\sigma_k^-(R)} \circ u_k^R - \varepsilon u_k^{\sigma_l^-(R)} \circ u_l^R$ where $k, l \in R$ such that $k-1, l-1 \notin R$ and $\varepsilon = -1$ if k and l are positive and 1 otherwise, along with $z_k^R = u_{k-1}^{\sigma_k^-(R)} \circ u_k^R$ where $k \in R$, but $k-1, k-2 \notin R$;

(ii) the maps $v_k^R = (-1)^{\kappa(R,k)} \cdot u_k^R$ for all $R \in C_{m,n}$ and for all $k \in R$ allowed, with relations generated by $(\rho'_{k,l})^R = v_l^{\sigma_k^-(R)} \circ v_k^R - v_k^{\sigma_l^-(R)} \circ v_l^R$ where $k, l \in R$ such that $k-1, l-1 \notin R$ along with $(z'_k)^R = v_{k-1}^{\sigma_k^-(R)} \circ v_k^R$ where $k \in R$, but $k-1, k-2 \notin R$;

(iii) the morphisms $w_k^R = \varepsilon(R,k)(u_k^R)$ for all $R \in C_{m,n}$ with $\varepsilon(R,k) = (-1)^{\nu(R,k)+\kappa_R}$ if $k < 0$ and $\varepsilon(R,k) = 1$ otherwise, for all $k \in R$ allowed, with relations generated by $(\rho''_{k,l})^R = w_l^{\sigma_k^-(R)} \circ w_k^R + w_k^{\sigma_l^-(R)} \circ w_l^R$ where $k, l \in R$ such that $k-1, l-1 \notin R$ along with $(z''_k)^R = w_{k-1}^{\sigma_k^-(R)} \circ w_k^R$ where $k \in R$, but $k-1, k-2 \notin R$.

Proof. The presentation with relations as per Item (i) follows directly from the previous discussion. The morphisms $\{v_k^R\}_{k,r}$ and $\{w_k^R\}_{k,r}$ still form generating sets for the morphisms differing from their u_k^R counterpart by a unit. Moreover, the zero relations $z_k^R, (z'_k)^R$ and $(z''_k)^R$ are also proportional for each R and k. We will now show that the same is true for the square relations. Without loss of generality we can assume that $k \leq l$. We first consider the relations from Item (ii). We write the terms of the relation as follows

$$\begin{aligned} v_l^{\sigma_k^-(R)} \circ v_k^R &= (-1)^{\kappa(R,k)+\kappa(\sigma_k^-(R),l)} \cdot u_l^{\sigma_k^-(R)} \circ u_k^R, \\ v_k^{\sigma_l^-(R)} \circ v_l^R &= (-1)^{\kappa(R,l)+\kappa(\sigma_l^-(R),k)} \cdot u_k^{\sigma_l^-(R)} \circ u_l^R. \end{aligned}$$

If k is negative, then

$$\kappa(R,k)+\kappa(\sigma_k^-(R),l)=\kappa(R,l)+\kappa(\sigma_l^-(R),k).$$

Hence we have

$$(\rho'_{k,l})^R = v_l^{\sigma_k^-(R)}\circ v_k^R - v_k^{\sigma_l^-(R)}\circ v_l^R = (-1)^{\kappa(R,k)+\kappa(\sigma_k^-(R),l)}\cdot \rho_{k,l}^R.$$

Otherwise, from the computation in Notation 5.2.13 we have $\kappa(R,k)=\kappa(\sigma_l^-(R),k)$ and $\kappa(R,l)=\kappa(\sigma_k^-(R),l)+1$ which implies

$$\begin{aligned}(\rho'_{k,l})^R &= v_l^{\sigma_k^-(R)}\circ v_k^R - v_k^{\sigma_l^-(R)}\circ v_l^R\\ &= (-1)^{\kappa(R,k)+\kappa(\sigma_k^-(R),l)}\cdot (u_l^{\sigma_k^-(R)}\circ u_k^R + u_k^{\sigma_l^-(R)}\circ u_l^R)\\ &= (-1)^{\kappa(R,k)+\kappa(\sigma_k^-(R),l)}\cdot \rho_{k,l}^R.\end{aligned}$$

For the square relations of Item (iii) we distinguish three cases. When $k\geq 0$ and $l\geq 0$ we have $\varepsilon(\sigma_l^-(R),k)=\varepsilon(R,l)=\varepsilon(\sigma_k^-(R),l)=\varepsilon(R,k)=1$ so $(\rho''_{k,l})^R=\rho_{k,l}^R$. Next, when $k<0\leq l$ we still have $\varepsilon(R,l)=1=\varepsilon(\sigma_k^-(R),l)$. However, by the computations in Notation 5.2.13 we have,

$$\varepsilon(\sigma_l^-(R),k)=(-1)^{\nu(\sigma_l^-(R),k)+\kappa_{\sigma_l^-(R)}}=(-1)^{\nu(R,k)+\kappa_R-1}=-\varepsilon(R,k),$$

so it holds that

$$\begin{aligned}(\rho''_{k,l})^R &= w_l^{\sigma_k^-(R)}\circ w_k^R + w_k^{\sigma_l^-(R)}\circ w_l^R\\ &= \varepsilon(R,k)\cdot u_l^{\sigma_k^-(R)}\circ u_k^R + \varepsilon(\sigma_l^-(R),k)\cdot u_k^{\sigma_l^-(R)}\circ u_l^R\\ &= \varepsilon(R,k)\cdot \rho_{k,l}^R.\end{aligned}$$

Finally, when $k<l<0$ we compute

$$\varepsilon(\sigma_l^-(R),k)=(-1)^{\nu(\sigma_l^-(R),k)+\kappa_{\sigma_l^-(R)}}=(-1)^{\nu(R,k)+\kappa_R}=\varepsilon(R,k)$$

and

$$\varepsilon(R,l)=(-1)^{\nu(R,l)+\kappa_R}=(-1)^{\nu(\sigma_k^-(R),l)+1+\kappa_{\sigma_k^-(R)}}=-\varepsilon(\sigma_k^-(R),l)$$

which yields, after similar computations

$$(\rho''_{k,l})^R = -\varepsilon(R,k)\varepsilon(R,l)\cdot \rho^R_{k,l}$$

and this completes the proof. □

5.3 Tilting object

In what follows, α denotes an element of $J_{m,n}$, *i.e.* a plain partition.

Proposition 5.3.1. *The subcategory* $\mathrm{Thick}(\{\mathcal{P}_\alpha|\alpha \in J_{m,n}\})$ *is the category of perfect complexes of* $J_{m,n}$ *which is equivalent to its bounded derived category.*

Proof. In the proof of the main result of Section 3 we constructed by induction each projective module as the end of a sequence of cones starting in $\mathcal{Y}_{m,n}$. The argument only relied on the fact that each projective had a quotient belonging to the family $(\mathcal{P}_\alpha)_\alpha$. Restricting the family to the modules associated with the plain partitions, *i.e.* those with the bar pushed all the way to the right, proves the claim about $\mathrm{Thick}(T)$. □

Using Notation 5.2.13 we set

$$T := \bigoplus_{\alpha\in J_{m,n}} \mathcal{P}_\alpha[\kappa_\alpha]. \tag{5.9}$$

The goal of the rest of this section is to prove that the complex T is tilting, meaning it has no self extension, and to compute the algebra $\mathrm{End}\, T^{op}$. We can hope to do so because we know the extensions only appear in one degree for each couple of summands of T. Shifting each summand of T by κ_α amounts to concentrate all the morphisms in degree zero. Note that there are subgraphs of Figure 5.1 for which such a thing is not possible.

Example 5.3.2. Take for instance, the cycle made of the objects $\mathcal{P}_{(2,2|)}$, $\mathcal{P}_{(1,1|)}$ and $\mathcal{P}_{(0|2)}$ in Figure 5.1. Say we shift $\mathcal{P}_{(2,2|)}$ by n. The extension from $\mathcal{P}_{(2,2|)}$ to $\mathcal{P}_{(1,1|)}$ forces the later to be shifted by $n+1$. In the same way, $\mathcal{P}_{(0,2|)}$ has to be shifted by $n+2$. Lastly, the morphism in degree zero from from $\mathcal{P}_{(0|2)}$ to $\mathcal{P}_{(2,2|)}$ means that the later should then be shifted by $n+2$ as well which is a contradiction. More generally cyclic path cannot be shifted in a way to make the morphisms be in degree zero unless all the morphisms were already concentrated in degree zero. Another subgraph that leads to contradicting shifts is the one whose vertices are $(11|)$, $(1|2)$, $(12|)$ and $(01|)$. The extensions along this path would require $\mathcal{P}_{(11|)}$ to be shifted by two compared to $\mathcal{P}_{(1|2)}$. But the morphism in degree zero between the two requires them to be shifted by the same amount. Please note that the composition of the morphisms along these paths is zero using the relations exhibited in the previous subsection.

Lemma 5.3.3. *The object T has no self extensions.*

Proof. First note that the quantity κ_α only depends on the non zero values of the partition α *i.e.* those indexed by S_α. Hence if there is a morphism in degree zero $\mathcal{P}_\alpha \to \mathcal{P}_\beta$ then $\kappa_\alpha = \kappa_\beta$. This follows from Item (iv) of Lemma 5.1.1 and the fact that we only look at plain partitions.Moreover, if J is allowed and $|J| = p$ then $\kappa_{q_J(\alpha)} = \kappa_\alpha + p$. We put these two remarks together. If there exists a non zero morphism

$$\mathcal{P}_\alpha[\kappa_\alpha] \to \mathcal{P}_\beta[\kappa_\beta][p] \tag{5.10}$$

by Proposition 5.1.12, there exists a unique subset J of S_α such that $q_J(\alpha) \in [f(\beta), \beta]$ and $|J| = -\kappa_\alpha + \kappa_\beta + p$. Because J is allowed for α, $\kappa_\alpha = \kappa_{q_J(\alpha)} - |J|$. Because there is a non zero morphism of modules from $\mathcal{P}_{q_J(\alpha)}$ to $\mathcal{P}_\beta$, $\kappa_\beta = \kappa_{q_J(\alpha)}$. Thus, $|J| = \kappa_\beta - \kappa_\alpha$, leading to $p = 0$ and T has no self extensions. $\square$

Proposition 5.3.1 together with Lemma 5.3.3 show that the objet T is tilting. We now describe its algebra of endomorphisms and to do so we recall the construction of higher Auslander algebras of type A following convention from [32, Definition 2.12]. Note that we compose arrows using a different convention but everything else is written as close to that source as possible. Let d and s be integers. The *higher Auslander algebra of type A_s^d* is constructed as a bound quiver algebra. The underlying set Q_0 of the quiver Q is the set of increasing sequences of length $d+1$ with values in $\{1, \dots, d+s\}$. Let $x = (x_0, \dots, x_d)$ be an element of Q_0. We use the symbol $\in$ to indicate that a value appears in x. For $k \in x$, we define a partial transformation σ_k^+ on Q_0 by

$$\sigma_k^+(x) = (x_0 < \dots x_i < k+1 < x_{i+2} < \dots x_d)$$

whenever the resulting sequence is increasing *i.e.* $x_{i+2} \neq k+1$. Similarly we write σ_k^- for the partial map that replaces k by $k-1$ in the sequence, whenever possible. Let the set Q_1 of arrows of the quiver consist of elements α_k^x with source x and target $\sigma_k^+(x)$ whenever the target is well defined. In the path algebra of the resulting quiver $\Bbbk Q$ we consider the ideal $I = \langle G \rangle$ generated by the vector space G with basis the following combinations of paths of length two

$$\rho^x_{k,l} = \begin{cases} \alpha_k^{\sigma_l^+(x)}\alpha_l^x - \alpha_l^{\sigma_k^+(x)}\alpha_k^x & \text{if } k,l \in x \text{ and } k+1, l+1 \notin x, \\ \alpha_k^{\sigma_{k+1}^+(x)}\alpha_{k+1}^x & \text{if } l = k+1 \in x \text{ and } l+1 \notin x. \end{cases}$$

Then set $(\Bbbk Q)/I)^{op}$ to be the higher Auslander algebra A^d_s. With these relations and the usual grading by length of paths it is clear that A^d_s is quadratic. Its quadratic dual is

$$A^! = (\Bbbk Q^{op}/\langle G^\perp \rangle)^{op}$$

where $\Bbbk Q^{op} = \Bbbk(Q^{op})$ and $G^\perp$ is the orthogonal complement of G in the dual of $\Bbbk Q_2$, the vector space with basis the paths of length two in the quiver Q [45]. It remains to compute the orthogonal of G in $\Bbbk Q_2$ to get a presentation of the quadratic dual as a quiver with relations.

Proposition 5.3.4. *The orthogonal component $G^\perp$ of G has basis*

$$\begin{cases} (\alpha_k^x)^{op}(\alpha_{k+1}^{\sigma_k^+(x)})^{op} \\ (\alpha_k^x)^{op}(\alpha_l^{\sigma_k^+(x)})^{op} + (\alpha_l^x)^{op}(\alpha_k^{\sigma_l^+(x)})^{op} \end{cases}$$

where $k, l \in x$ while $k+1, l+1 \notin x$.

Proof. It is clear that these elements are in $G^\perp$. We know that

$$\dim G + \dim G^\perp = \dim \Bbbk Q_2.$$

We argue that the set above is precisely a basis for $G^\perp$ for reasons of cardinality. To do so, notice that the composition of two arrows in the quiver modifies either one or two elements of the sequence. The case where it modifies one element corresponds to the first relations in the equation above. When two elements are modified there are two cases. Either the order matters or it does not. If it does then we are in the case of the zero relations of A^d_s. If it doesn't, than the 2-path appears in the commutation relation of A^d_s but also in the anti commutation relation we exhibited for $(A^d_s)^!$ just now. Hence there is a partition of a basis of $\Bbbk Q_2$ into the paths in G and $G^\perp$. □

Note that in the quadratic dual, squares commute with a sign *i.e.* $ab + cd = 0$. What remains to prove for Theorem E is that the signs of the squares can be modulated meaning that we can construct isomorphisms between the quiver algebra modulo relations

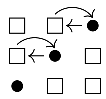


Figure 5.9: illustrating the duality of the relations

with $ab + cd = 0$ and the one with $ab - cd = 0$ for some square relations in the ideal I. This is not true in general but holds for certain configurations of squares in $J_{m,n}$ and A_s^d.

Proof of Theorem E. The restrictions of the presentation of $\mathcal{Y}_{m,n}$ given in 5.2.14 Item (ii) to the indecomposable components of the object T correspond exactly to the presentation of A_{m+1}^{n-1} after replacing the configurations by their complements and shifting them by $+m$. $\square$

Proposition 5.3.5. *The algebra $J_{m,n}$ is derived equivalent to $(A_{n+1}^{m-1})^!$.*

Proof. The generators and relations of the algebra $\mathrm{End}(T)^{op}$ given in 5.2.14 Item (iii) after restricting to the tilting object are exactly the relations of the quadratic dual of A_{n+1}^{m-1} described in Proposition 5.3.4 after shifting the values of the configurations by $+m$. $\square$

As a direct corollary of the proof we have the following nice result which is probably known but for which I have not found suitable reference in the literature.

Corollary 5.3.6. *There is an isomorphism of algebras between A_s^d and the quadratic dual of A_{d+2}^{s-2}.*

5.4 Interlacement

We want to deduce a description of the quadratic dual of the higher Auslander algebras using interlacing sequences and a sign rule in a way that mirrors a known description for higher Auslander algebras of type A introduced in [47]. We also want a description of $\mathcal{Y}_{m,n}$ using interlacement. Let $x = (x_0, \dots, x_d)$ and $y = (y_0, \dots, y_d)$ be two integer sequences. Still following convention from [31] we say that x *left interlaces* y when

$$x_0 \leq y_0 < x_1 \leq y_1 < \cdots < x_d \leq y_d \tag{5.11}$$

and write it $x \preccurlyeq y$. Similarly, we say that x *right interlaces* y when

$$x_0 < y_0 \leq x_1 < y_1 \leq \cdots \leq x_d < y_d \tag{5.12}$$

and write it $x \preccurlyeq y$. In [47], the authors use these notions to describe the higher Auslander algebras as well as certain morphisms spaces in the following way. To paraphrase [31], for each sequence x as above, consider the indecomposable A_n^{d-1}-module M_x with vector space $\Bbbk$ for sequences y with which x left interlaces and zero everywhere else. Arrows $\Bbbk \to \Bbbk$ act as the identity and the rest act as zero. Denote by M the module

$$\bigoplus_x M_x. \tag{5.13}$$

Theorem 5.4.1 ([47, Section 3])**.** *The module M is a cluster tilting object for A_n^{d-1} and $A_n^d \cong \operatorname{End}(M)$.*

Moreover, there is an explicit description of this endomorphism algebra which allows us to say that A_n^d is the incidence algebra of the left interlacement relation.

Theorem 5.4.2. *For two sequences x, y we have*

$$\dim_\Bbbk \operatorname{Hom}_{A_n^{d-1}}(M_x, M_y) = \begin{cases} 1 & \text{if } x \preccurlyeq y \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

Here is a new formulation using a convenient category.

Definition 5.4.3. Let n and d be integers. Define the positive interlacing category category $\mathcal{I}_{d,n}^+$ as follows:

- set $\operatorname{Ob}(\mathcal{I}_{d,n}^+) = \{$increasing sequences of length $d+1$ in $[\![0, d+n]\!]\}$;
- given two increasing sequences a and b in $\operatorname{Ob}(\mathcal{I}_{d,n}^+)$, set $\mathcal{I}_{d,n}^+(a,b)$ to be the vector space with basis $m_{a,b}$ if a and b interlace and zero otherwise.
- Given two composable morphisms $m_{a,b}$ and $m_{b,c}$, we define the composition in the following predictable manner

$$m_{b,c} \circ m_{a,b} = \begin{cases} m_{a,c} & \text{if } a \text{ and } c \text{ interlace,} \\ 0 & \text{otherwise.} \end{cases}$$

The morphism $m_{a,a}$ is the unit for each object a and it is easy to check that the composition law is associative.

Proposition 5.4.4. *There is an equivalence of categories A_n^d-Mod $\cong \operatorname{Fun}(\mathcal{I}_{d,n}^+, \Bbbk\text{-Mod})$*

Proof. The object

$$\bigoplus_x \mathrm{Hom}_{\mathcal{I}^+_{m,n}}(x, ?) \tag{5.15}$$

is a progenerator for $\mathrm{Fun}(\mathcal{I}^+_{m,n}, \Bbbk\text{-mod})$. $\square$

By Corollary 5.3.6, this immediately gives a description of the quadratic dual $(A^{n-2}_{d+2})^\dagger$.

Corollary 5.4.5. *There is an equivalence of categories* $(A^{n-2}_{d+2})^\dagger\text{-Mod} \cong \mathrm{Fun}(\mathcal{I}^+_{d,n}, \Bbbk\text{-Mod})$

We could also define a new category which describes more closely the combinatorics of $(A^{n-2}_{d+2})^\dagger$. For a sequence a of length d in $[\![0, d+n]\!]$ we denote by a^c the increasing sequence of length n consisting of the elements of $[\![0, d+n]\!]$ which do not appear in a.

Definition 5.4.6. Let n and d be integers. Define the signed interlacing category $\mathcal{I}^-_{d,n}$ as follows:

- set $\mathrm{Ob}(\mathcal{I}^-_{d,n}) = \{\text{increasing sequences of length } d+1 \text{ in } [\![0, d+n]\!]\}$;
- given two increasing sequences a and b in $\mathrm{Ob}(\mathcal{I}^+_{d,n})$, set

$$\mathcal{I}^-_{d,n}(a,b) = \begin{cases} \Bbbk \cdot m^-_{a,b} & \text{if } a^c \text{ and } b^c \text{ interlace,} \\ 0 & \text{otherwise.} \end{cases}$$

To define composition to match the signs of the quadratic dual note that morphisms are characterised by the set of indices at which complements of the source and the target differ. Given two composable morphisms $m^-_{a,b}$ and $m^-_{b,c}$ associated to sets $J_{a,b}$ and $J_{b,c}$, we define the composition in the following way

$$m^-_{b,c} \circ m^-_{a,b} = \begin{cases} (-1)^\epsilon m^-_{a,c} & \text{if } a^c \text{ and } c^c \text{ interlace,} \\ 0 & \text{otherwise.} \end{cases}$$

and ϵ is determined by an extra rule. Consider the following sets.

$$I = \{i \in [\![1, m]\!] | b_i \neq c_i\}$$
$$J = \{i \in [\![1, m]\!] | a_i \neq b_i\}$$

Then we set $\epsilon = (-1)^{|I|_J}$.

Claim 5.4.7. *There is an equivalence of categories* $(A^d_n)^\dagger\text{-Mod} \cong \mathrm{Fun}(\mathcal{I}^-_{d,n}, \Bbbk\text{-Mod})$

The reason we introduced $\mathcal{I}^-_{d,n}$ as a category was to enable us to discuss signs when composing arrows. Because we know Proposition 5.4.4 to be true, we could prove this statement indirectly. A direct proof would require more work. Instead of proving this, we will state and prove in a direct way a statement about $\mathcal{Y}_{m,n}$. Heuristically, the statement about $\mathcal{Y}_{m,n}$ should be less neat because, in that category, the sequences wrap around at $n+1$. The author thinks it gives an interesting expansion of Higher Auslander Algebras but is unsure of how to make sens of it. We revisite the order relation we introduced in 5.2. This time, we give a definition independent of any sequence of transformations linking the two objects.

Definition 5.4.8 (order $<^S_R$ associated to a pair of configurations (R,S))**.** Let R , S be two configurations. Take x to be the least value of $S^c \cap [\![0,n]\!]$ with regards to the naive order. If no such value exists, then take $x = n$. Set k^S_R so be the first value of $\mathcal{Z}_{m,n}$ which is to the left of x and not in R. The order relation associated to (R,S), denoted $<^S_R$ is thus

$$k^S_R <^S_R k^S_R + 1 <^S_R \cdots <^S_R n <^S_R -m <^S_R \cdots <^S_R k_R - 1.$$

We can use the order relation above to characterise morphisms in $\mathcal{Y}_{m,n}$ in terms of interlacings.

Proposition 5.4.9. *Let R and S be configurations and $k = k^S_R$ as above. Let a, respectively b, be the sequences obtained by taking the complement of R, respectively S, in $\mathcal{Z}_{m,n}$ and numbering the elements of the set using the order relation $<^S_R$. Then there exists a non zero morphism in $\mathcal{Y}_{m,n}$ from $\mathcal{P}_R$ to $\mathcal{P}_S[i]$ for some integer i if and only if $a \preccurlyeq b$.*

The proof of the proposition builds upon arguments that were already used in the previous section, so we only sketch it. The following notation is convenient for the sketch.

Notation 5.4.10. Let $\alpha = (\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r} | n^{\mu_{r+1}})$ be a partition and let J be an allowed subset of S_α We cut the set J into disjoint *maximal* blocks $B_1, \ldots, B_N$ of *consecutive indices* that point to *consecutive values* i.e. $B_i = \{j^i_1, \ldots, j^i_{N_i}\}$ with

$$\lambda_{j^i_k} = \lambda_{j^i_1} + k - 1$$

when $1 \leq k \leq N_i$. Blocks can be of size $N_i = 1$, see Example 5.4.11.

Example 5.4.11. Consider the partition $\alpha = (0,1,2,3,5,7,8,9)$ and the subset $J = \{2,3,5,6,7\}$. We describe the blocs of J. The first one is $B_1 = \{2,3\}$ associated to values $1,2$. The second one is $B_2 = \{5\}$ associated to value 5. The third one is $B_3 = \{6,7\}$ associated to values $7,8$.

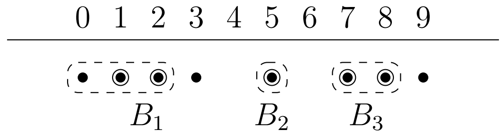


Figure 5.10: The configuration of Example 5.4.11

Sketch of proof of 5.4.9. First we assume that $a \preccurlyeq b$. We set

$$J = \bigsqcup_{\substack{i=0 \\ x_i \leq n}}^{d} \rrbracket x_i, \min(y_i, n) \rrbracket. \tag{5.16}$$

This subset of S_α is well defined as each interval $\rrbracket x_i, \min(y_i, n)\rrbracket$ is made up of values that appear in α because the sequences interlace and their elements are gaps in the configurations of the corresponding partitions. It is allowed because for each interval x_i is a gap in α. Hence there is a non zero extension from α to $q_J(\alpha)$. There is a degree zero morphism from $\mathcal{P}_{q_J(\alpha)}$ to $\mathcal{P}_\beta$ by Item (v) of Proposition 5.1.1. It remains to see that the composition of the two is non zero by checking the last assumption of Proposition 5.1.11 like in the proof of 5.2.10.

Conversely, assume we have a non zero morphism from $\mathcal{P}_\alpha$ to $\mathcal{P}_\beta[i]$. Then by Proposition 5.1.12 there exists a unique subset J of S_α such that ϕ factors through $\mathcal{P}_{q_J(\alpha)}[|J|]$. The sequence a left interlaces with the sequence c associated to $q_J(\alpha)$. See Sections 2, 4

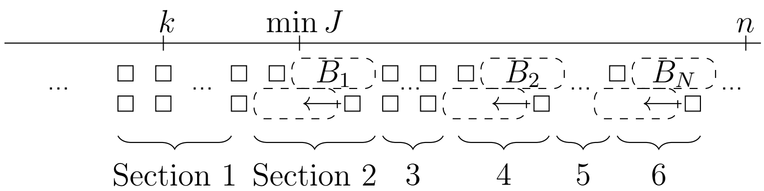


Figure 5.11: interlacing gaps in the positive side for α and $q_J(\alpha)$

and 6 of Figure 5.11. The rest of the values, starting at the index $N_N + 1$ of the sequences a and c match because α and $q_J(\alpha)$ share the same multiplicities. Note that in Figure 5.11, we represented a situation where $\varepsilon \notin J$. There are as many beads associated to each block as elements in that block. When $\varepsilon \in J$, it is the case for all the blocks except maybe the first one. Figure 5.12 illustrates how to think of the first block in that case. Notice how $k = \min J$. It is clear that a and b interlace in that situation as well. Next, by Proposition 5.1.1 Item (v) as well as Theorem 3.3.4, the abaci of $q_J(\alpha)$ and β match in columns k to $n-1$. Otherwise, there would also be a non trivial extension from $q_J(\alpha)$ to β. This corresponds to the first M entries of the sequences c and b. By Item (v) of

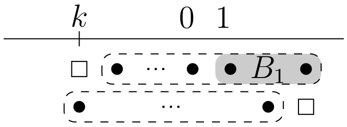


Figure 5.12: The first block and its effect when $\varepsilon \in J$

Proposition 5.1.1, we have the *interlacing* of the ending coefficients $y_i \leq x_i < y_i$. There exists k such that $y_i = c_k$ and $x_i = a_k$ by the argument made in the fourth paragraph of the proof of Lemma 5.2.10. Hence c left interlaces with b.

This results in a left interlacing with b. Please note that (left) interlacement is not in general transitive, however, in this specific case, the sequences a and c are the same starting from index N_N while c and b match before index M. We only have to check that "nothing bad happens" at the junction between these two sequences. The key is that $M \geq N_N$. $\square$

Having finished this characterisation it is possible to define an interlacing category like $\mathcal{I}^-_{d,n}$ or $\mathcal{I}^+_{d,n}$ with sign conventions matching any of the presentations of Corollary 5.2.14.

We conclude this section and this thesis by showing that the interlacing restricted to the indecomposable summands of tilting object T, is compatible with the interlacement describing the Higher Auslander Algebra. To do so, we need to find a minimum k which is compatible with all the sequences encountered in T simultaneously. This is possible on $\mathcal{Y}^{tilt}_{m,n}$, which we use to denote the full subcategory of $\mathcal{Y}_{m,n}$ whose objects are the indecomposable summands of T and their shifts, but not on $\mathcal{Y}_{m,n}$. The following arguments rely on elementary manipulations of sequences.

Suppose $a = (a_1, \ldots, a_{n+1})$ left interlaces with $b = (b_1, \ldots, b_{n+1})$ according to the order relation $<^S_R$ whose minimum is k^S_R. Then the maximum for that order is $k^S_R - 1$. If $a_n \leq^S_R b_n <^S_R k^S_R - 1$ then the sequences a and b also left interlace for the order relation with minimum $k^S_R - 1$. Suppose we have the equality $a_{n+1} \leq^S_R k^S_R = b_{n+1}$ and consider the sequences $a' = (a_{n+1}, a_1, \ldots, a_n)$ and $b' = (b_{n+1}, b_1, \ldots, b_n)$. Then a' left interlaces with b' for the order relation with minimum a_n. Reciprocally, if a' left interlaces with b' for that new order, then a left interlaces with b for the old one. At the same time, notice that the sequences associated to elements of the full subcategory $\mathcal{Y}^{tilt}_{m,n}$ all contain the value $-m$ and that for two sequences that interlace, the value $-m$ must be present at the same index. Combining the two procedures above and this last remark we get that choosing $-m$ as the minimum of our total order and numbering the elements of the complements of the configurations according to that order gives us sequences that interlace if and only if there is a non zero morphism between the corresponding objects.

The bijection between the indecomposable summands of the tilting object and the vertices of the quiver of the Higher Auslander algebra A^{n-1}_{m+1} is given by $(a_1, a_2, \dots, a_{n+1}) \mapsto (a_2 + m, \dots, a_{n+1} + m)$ because $a_1 = -m$. The bijection with the vertices of the quadratic dual of the Higher Auslander algebra $(A^{m-1}_{n+1})^{\dagger}$ is given by numbering the elements of the configurations from $-m$ upward, knowing that $-m$ never appears in the configurations of $\mathcal{Y}^{tilt}_{m,n}$, and then translating them by $+m$ in the same way.

Example 5.4.12. The configuration of the partition $(0, 1, 2, 3, 3, 7, 8, 9)$ in $J_{8,9}$ is

$$\{-4, 0, 1, 2, 3, 7, 8, 9\}.$$

The complement of this configuration, ordered from -8 is the sequence

$$(-8, -7, -6, -5, -3, -2, -1, 4, 5, 6).$$

Its image in (A^{n-1}_{m+1}) is $(1, 2, 3, 5, 6, 7, 12, 13, 14)$. Its image in $(A^{m-1}_{n+1})^{\dagger}$ is

$$(4, 8, 9, 10, 11, 15, 16, 18).$$

Appendix A

Two more figures

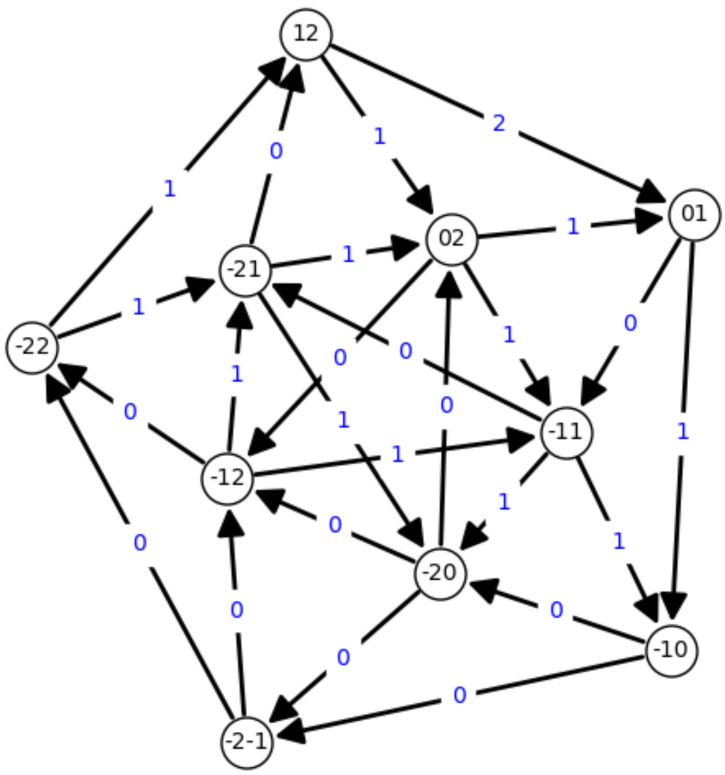


Figure A.1: A relabeling of Figure 5.1 using configurations

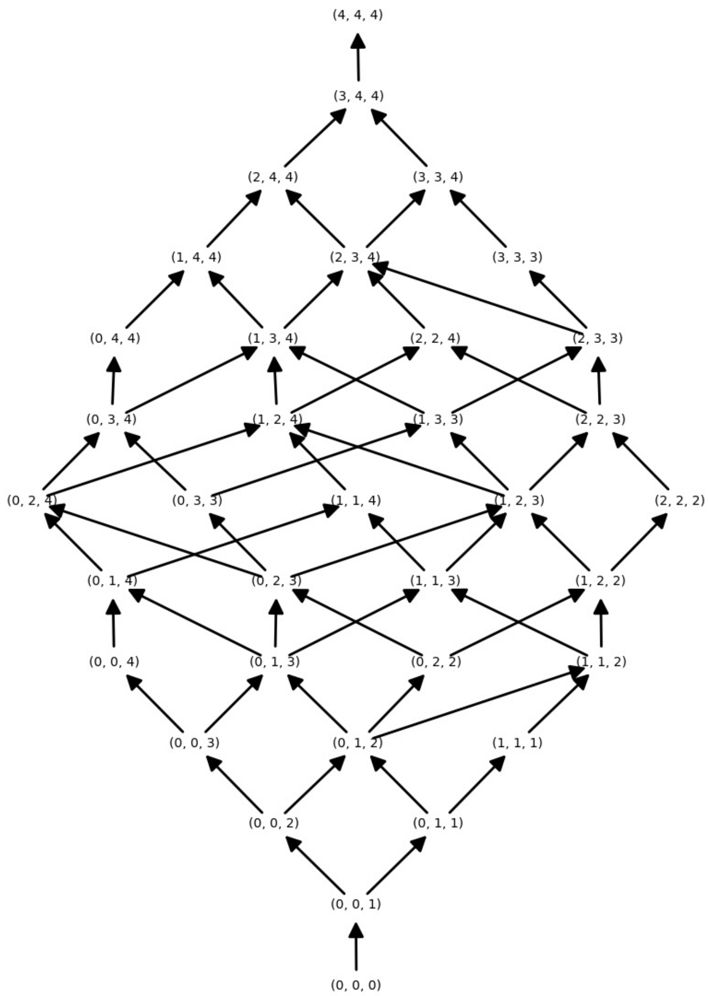


Figure A.2: The Hasse diagram of J_{34}

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