Fractionally Calabi-Yau Lattices

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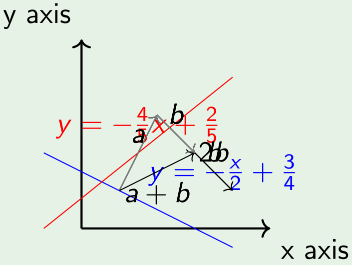
Linear algebra

Algebra in school : computations using $\times, +$

- Solving systems of equations
- describing affine lines
- vectors
- solving polynomial equations ... but this is not linear anymore

Example

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$



Matrices

All this can be described using *matrices*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 4 \\ 0 & 3 & 1 & 2 & 2 \end{pmatrix}$$

This is a 4 by 5 matrix.

Matrix operations

- Sum : termwise (a_{ij} adds with b_{ij}).
- Product : more complicated, but (m by n) times (n by p) gives a (m by p) matrix.
- Outer product : scale everything by some scalar α : a_{ij} becomes $\alpha \times a_{ij}$

Special cases

- (m by m) matrices : square matrices. Closed under multiplication. They form an *algebra*.
- (m by 1) matrices : column vectors.
- (1 by n) matrices : row vectors.
- (m by m) matrices act on the left of column vectors (m by 1).
- (n by n) matrices act on the right of row vectors (1 by n).

Example

A set of linear equations can now be written with matrices as follows.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \quad \rightsquigarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

Subspaces of square matrices

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \supseteq \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ & \ddots & \vdots \\ \text{\huge 0} & & a_{mm} \end{pmatrix} \supseteq \begin{pmatrix} 0 & a_{12} \cdots & a_{1m} \\ & \ddots & \vdots \\ & & a_{(m-1)m} \\ \text{\huge 0} & & 0 \end{pmatrix}$$

Example

$$\left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right\} \curvearrowright \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} \qquad \left\{ \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

This is what I study.

Partially ordered sets (posets)

Definition

Let P be set and let $\leq$ be a binary relation on P. We say that P equipped $\leq$ is a *partially ordered set* (or poset) if $\leq$ is

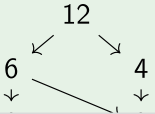
- reflexive : for all x in P, it holds that $x \leq x$;
- antisymmetric : for all x, y in P, if $x \leq y$ and $y \leq x$ then $x = y$;
- transitive : for all x, y, z in P, if $x \leq y$ and $y \leq z$ then $x \leq z$;

Example

$[1 \leq 2 \cdots \leq n] \subset \mathbb{Z} \subset \mathbb{R}$ (total orders)

Example

Consider $D = \{1, 2, 3, 4, 6, 12\}$ and the relation $x \prec y$ if x devides y.



The incidence algebra

Let $\Bbbk$ be a field and P a finite poset with relation $\leq$.

Definition (The incidence algebra $A = A_{\Bbbk}(P)$...)

... is the $\Bbbk$-vector space with basis (x, y) such that $x \leq y$. Multiplication :

$$(x,y)(z,t) = \begin{cases} (x,t) & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

Primitive idempotent : $e_x = (x, x)$. P is finite $\rightsquigarrow$ unit : $1_A = \sum_{x \in P} e_x$.

Example (divisors of 6)

$A_{\Bbbk}(D)$ has basis
$\{e_1, e_2, e_3, e_6, (1,2), (1,3), (1,6), \ldots\}$

Also described with matrices !

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{24} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

Lattices

Definition

If for all pair x, y of elements of P there exists

- a least upper bound, denoted $x \vee y$ (read "x *join* y") ;
- a greatest lower bound, denoted $x \wedge y$ (read "x *meet* y") ;

then we say that P is a *lattice*

Example (divisors of 12)

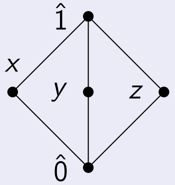
It is a lattice : meet $\leftrightarrow$ greatest common divisor (gcd). join $\leftrightarrow$ least common multiple (lcm)

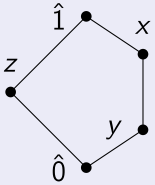
From now on P is a lattice and we use the letter L. It has a maximal element $\hat{1}$ and minimal element $\hat{0}$.

Distributivity

A lattice is *distributive* if the join distributes over the meet and vice versa.

Two theorems of Birkhoff





An subset I of a poset P is an *order ideal* if for all $x \in I$ and $y \in P$, when $y \leq x$ then $y \in I$.
$\rightsquigarrow$ Form a distributive lattice with $\cup$ and $\cap$

1. *A lattice is distributive iff it does not contain a sublattice isomorphic to one of the above.*

2. *A finite lattice is distributive if and only if it is isomorphic to the lattice of order ideals of its poset of join irreducible elements.*

Definition (Less rigidity)

A lattice is

- *join semidistributive* if when $x, y, z \in X$ satisfy $x \vee y = x \vee z$, we have $x \vee (y \wedge z) = x \vee y$.
- *meet semidistributive* if when $x, y, z \in X$ satisfy $x \wedge y = x \wedge z$, we have $x \wedge (y \vee z) = x \wedge y$.
- *semidistributive* if it is both join and meet semidistributive.

Definition (More regidity)

A distributive lattice L is *boolean* if its elements have a unique *complement*.

Example

- Semidistributive : Lattices of torsion classes of a path algebra.
- Distributive : the main example of this thesis
- Boolean : power sets of finite sets.

Dimensions

- Maximal covering number : the maximal integer k such that there exists $x \in P$ which covers exactly k elements. Denote it $\mathrm{cov}(P)$.
- Order dimension : Minimal integer t such that there exists t linear extensions of $\leq$ whose intersection is equal to $\leq$.

Classical theorems

- Kelly et Trotter 1982 Let t be the order dimension of P. Then t is the minimal integer such that P embeds as a poset in the product of chains $\mathbb{R}^t$ equipped with term-wise comparison
- Dilworth 1950If L be a finite distributive lattice then $\mathrm{OrdDim}(L) = \mathrm{cov}(L)$.

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Antichain modules

Definition (Antichain)

A subset C of a poset is an antichain if it contains pairwise incomparable elements.

Notation : for $\alpha \in L$, let P_α, resp. S_α, denote the projective indecomposable, resp. simple, module associated to α.

Core idea for representation of posets(Iyama et Marczinzik 2022, Proposition 2.1)

antichains C below $\alpha \leftrightarrow$ submodules $P_\alpha \leftrightarrow$ modules M_C^α with top S_α
M_C^α is called an *antichain module*. For lattices : we get a *antichian projective resolution* $\mathcal{P}_C^\alpha$ in a natural way.

$$0 \to P_{\wedge C} \to \cdots \to \bigoplus_{\substack{S \subseteq C \\ |S|=k}} P_{\wedge S} \xrightarrow{\partial_k} \cdots \to \bigoplus_{s \in C} P_s \to P_\alpha \to M_C^\alpha \to 0$$

About the boundary maps

To define a ∂_k, fix an arbitrary ordering of the antichain $C = \{c_1 < \cdots < c_r\}$. Consider a subset $S = \{s_1 < \cdots < s_k\}$ of C. Then set

$$\partial_k(e_{\wedge S}) = \sum_{i=1}^{k} (-1)^i \cdot e_{\wedge (S - \{s_i\})} \tag{1}$$

Rigid antichains

Problem

These antichain resolutions are not always minimal.

Definition

Inclusive antichain For all subsets S and S' of C, if $\wedge S \leq \wedge S'$ then $S' \subseteq S$

Intersective antichain For all subsets S and S' of C, we have $(\wedge S) \vee (\wedge S') = \wedge(S \cap S')$.

Strong antichain For all S, S' subsets of C of same cardinal, $\wedge S$ and $\wedge S'$ are incomparable *i.e* if $\wedge S \leq \wedge S'$ then $S = S'$.

Boolean antichain C is both Inclusive and intersective.

Examples

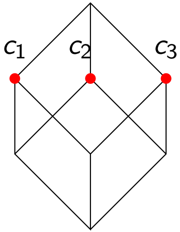

Figure – Boolean antichain

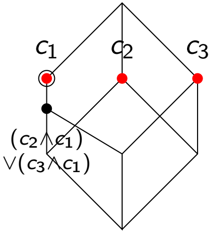

Figure – Strong, not intersective, antichain

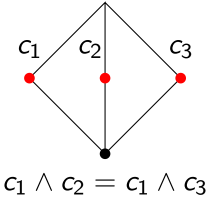

Figure – Antichain which is neither

Figure – Examples and non examples for key properties of antichains

Reduction

Lemma

An antichain is strong if and only if it is inclusive.

Claim

If an antichain below α does not contain α and is intersective then it is strong.

Corollary of the claim

An antichain below α which does not contain α is intersective is and only if it is boolean.

Lemma

An antichain is boolean if there is an anti lattice isomorphism between the sublattice of L spanned by C and α and the lattice $\mathcal{P}(C)$.

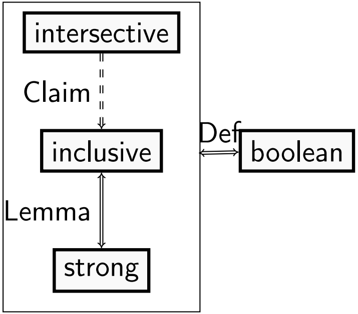

Figure – Properties of antichains and how they relate to one another

First results

Theorem (Iyama et Marczinzik 2022)

Let L be a finite join distributive lattice with at least two elements. The global dimension of the incidence algebra of L is the order dimemsion of L.

Theorem (Corollary 3.2.2)

Let L be a finite join semidistributive lattice with at least two elements. The global dimension of the incidence algebra of L is $\mathrm{cov}(L)$*.*

Ingredients :

Theorem 3.2.1 Let L be a finite join semidistributive lattice and $\mathbb{k}$ a field. Then the simple modules over $A_{\mathbb{k}}(L)$ are strong antichain modules.

Proposition 3.2.4 Antichain resolutions of strong antichains are minimal.

First results

Example (Tamari lattices)

-> the set of binary trees with $n+1$ leaves and order generated by tree. They are semidistributive lattices (Urquhart 1978 or Geyer 1994).

$$\mathrm{gldim}(T_n) = \max_{x \in T_n}(\mathrm{pdim}\, S_x) = \mathrm{cov}(T_n) = n-1.$$

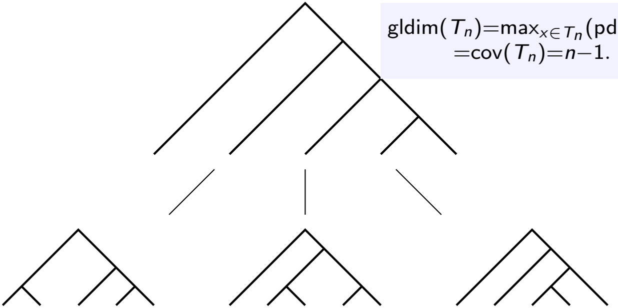

Figure – Covering relations from the maximum of Tamari 4

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Notation

- recall L is a finite lattice and A is its incidence algebra ;
- A a finite dimensional $\mathbb{k}$-algebra with finite global dimension ;
- $DA = \mathrm{Hom}_{\mathbb{k}}(A, \mathbb{k})$;
- $D^b(L) = D^b(A\text{-mod})$ the *bounded derived category* ;
- it is *triangulated*. Denote by [1] its suspension functor ;
- it has Serre functor $\mathbb{S} \cong - \otimes^{\mathbb{L}} DA$;
- recall : a triangulated category with Serre functor $\mathbb{S}$ is $\frac{d}{l}$-Calabi-Yau if $\mathbb{S}^l \cong [d]$ where d, l are integers.

Chapoton's Conjecture (F. Chapoton 2023)

Product formula

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of non negative integers. Assume that for all integer n there exists integers $m, D, d_1, \ldots, d_m$ satisfying the *product formula* $s_n = \prod_{i=1}^{m} \frac{D-d_i}{d_i}$.

Example

$$\binom{m+n}{m} = \frac{m+n}{1} \times \cdots \times \frac{n+1}{m}.$$

Representation theoretic conjecture

Then there should exist a family of posets $(P_n)_{n\in N}$ with cardinals $|P_n| = s_n$ such that the bounded derived category $D^b(P_n)$ is fractionally Calabi-Yau of dimension $\frac{C}{D}$ where $C = \sum_{i=1}^{m} D - 2d_i$.

Links to geometry

Previous progress

Frédéric Chapoton 2007

On the Grothendieck group $K_0(D(Tam_n))$ the serre functor satisfies $\mathbb{S}^{2n+2} = id_{k_0}$.

Rognerud 2021, Theorem 8.3

Let $n \in \mathbb{N}$. The bounded derived category $D^b(Tam_n)$ is $\frac{n(n-1)}{2n+2}$ fractionally Calabi-Yau.

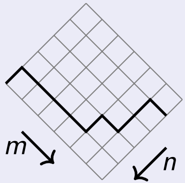
Rognerud 2021, Theorem 3.1

Let P be a finite poset with a unique minimal element or a unique maximal element. If there are integers d and l such that $\mathbb{S}^l(P_x) \cong P_x[d]$ in $D^b(P)$ for all projective indecomposable modules of P, then the category $D^b(P)$ is $\frac{d}{l}$-Calabi–Yau.

The lattice of order ideals of the product of two chains

Denote it $J_{m,n}$. There is an isomorphism : $J_{m,n} \xleftrightarrow{\sim} \{(a_1, \dots, a_m) | \text{non decreasing in} [\![0, n]\!]\}$.

$$\Rightarrow |J_{m,n}| = \binom{m+n}{m} = \frac{m+n}{1} \times \cdots \times \frac{n+1}{m}.$$



Yldrm 2019, Theorem 4.1

The functor $\mathbb{S}$ has finite order on the Grothendieck group $K_0(D^b(J_{m,n}))$. Specifically, $\mathbb{S}^{m+n+1} = \pm id_{K_0}$.

G. Preprint, (Theorem C)

Let $m, n \in \mathbb{N}$. The bounded derived category $D^b(J(P_{m,n}))$ is $\frac{mn}{m+n+1}$-fractionally Calabi-Yau.

Theorem (3.5.1)

Let L be a finite lattice, let d and l be integers. Suppose there exists a family of antichains $(C_\alpha)_{\alpha \in L}$ of L such that for all $\alpha \in L$, the following assumptions hold.

1. *The antichain C_α is below α.*
2. *The module M_C^α is non zero and there is an isomorphism*
$$\mathbb{S}^l M_{C_\alpha}^\alpha[0] \cong M_{C_\alpha}^\alpha[d]$$
in $D^b(L)$.
3. *The antichain C_α is strong.*

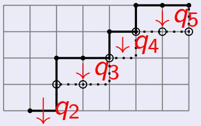
Then $D^b(L)$ is $\frac{d}{l}$-Calabi–Yau.

Sketch of proof of Theorem C

Invtervals of Yıldırım's

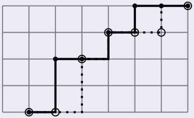
For each $\alpha \in J_{m,n}$ we construct an antichain C_α. Denote $\mathcal{P}_\alpha$ its object in $D^b(J_{m,n})$.

$C_\alpha = \{q_i(\alpha) | i \in S_\alpha\}$



$\rightsquigarrow$ The associated antichain module is an interval with minimum $f(\alpha)$Yldrm 2019, Proposition 2.13 $\rightsquigarrow$

The bounds of the interval $[f(\alpha), \alpha]$



- $\mathbb{S}^{m+n+1}(\mathcal{P}_\alpha) \cong \mathcal{P}_\alpha[mn]$. Yldrm 2019, Proposition 2.13 + rewriting in the derived category.
- The antichain C_α is boolean and in particular strong.

$\boxed{\Rightarrow \text{Apply Theorem 3.5.1.}}$

Sketch of proof of Theorem 3.5.1

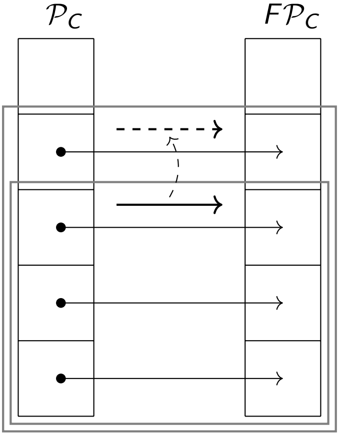


Figure – How we get it

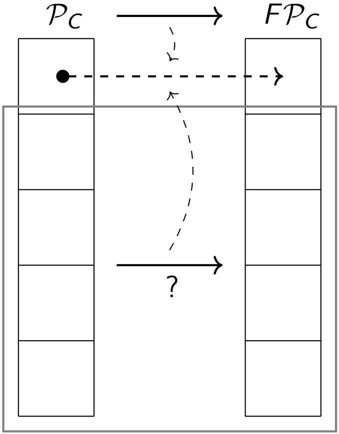


Figure – What we want

Sketch of proof of Theorem 3.5.1

We suppose (C_α) are strong antichains and $\mathbb{S}^n \mathcal{P}_C^\alpha \cong \mathcal{P}_C^\alpha[m]$.
Goal : show that $\mathbb{S}^n P_C^\alpha \cong P_C^\alpha[m]$ and apply Rognerud 2021, Theorem 3.1.

- Induction on α ("*outer induction*").
- truncations triangles and 2 out of three lemma :

$$\begin{tikzcd}
\mathcal{P}_C^\alpha \arrow[r] \arrow[d, "\wr"] & \sigma_{\geq 1}\mathcal{P}_C^\alpha \arrow[r] \arrow[d, dashed, "\exists ?"] & P_\alpha[1] \arrow[r] & \mathcal{P}_C^\alpha[1] \arrow[d, "\wr"] \\
F(\mathcal{P}_C^\alpha) \arrow[r] & F(\sigma_{\geq 1}\mathcal{P}_C^\alpha) \arrow[r] & F(P_\alpha[1]) \arrow[r] & F(\mathcal{P}_C^\alpha)[1]
\end{tikzcd}$$

- "*inner induction*" : Show that $\mathbb{S}^n \sigma_{\geq i}\mathcal{P}_C^\alpha \cong \sigma_{\geq i}\mathcal{P}_C^\alpha[m]$ with the triangle :

$$\sigma_{\geq i}\mathcal{P}_C^\alpha \xrightarrow{\partial_i[i]} (\mathcal{P}_C^\alpha)^{i-1}[i] \longrightarrow \sigma_{\geq i-1}\mathcal{P}_C^\alpha[1] \longrightarrow \sigma_{\geq i}\mathcal{P}_C^\alpha[1]$$

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Higher Auslander Algebras of type A

Definition (Iyama 2007)

A finite dimensional algebra Γ is *n-Auslander algebra* if it satisfies $\operatorname{gldim}\Gamma \leq n+1 \leq \operatorname{domDim}\Gamma$.

Consider the quiver $Q = (Q_0, Q_1)$ defined by :

- Q_0 : increasing sequences of length $d+1$ with values in $\{1, \dots, d+s\}$.
- Q_1 : arrows $\alpha_k^x : x \to \sigma_k^+(x)$ where σ_k^+ *is a partial map* on Q_0.

and ideal $I = \langle G \rangle$ generated by

$$\rho_{k,l}^x = \begin{cases} \alpha_k^{\sigma_l^+(x)} \alpha_l^x - \alpha_l^{\sigma_k^+(x)} \alpha_k^x & \text{if } k, l \in x \text{ and } k+1, l+1 \notin x, \\ \alpha_k^{\sigma_{k+1}^+(x)} \alpha_{k+1}^x & \text{if } l = k+1 \in x \text{ and } l+1 \notin x. \end{cases}$$

Then set $(\Bbbk Q)/I)^{op}$ to be the higher Auslander algebra A_s^d.

Their quadratic duals is

$$(A_s^d)^\dagger = (\mathbb{k}Q^{op}/\langle G^\perp\rangle)^{op}.$$

The orthogonal component $G^\perp$ of G has basis

$$\begin{cases}(\alpha_k^x)^{op}(\alpha_{k+1}^{\sigma_k^+(x)})^{op} \\ (\alpha_k^x)^{op}(\alpha_l^{\sigma_k^+(x)})^{op} + (\alpha_l^x)^{op}(\alpha_k^{\sigma_l^+(x)})^{op}.\end{cases}$$

Observations

- A_s^d and $(A_s^d)^{op}$ are isomorphic. Send $x_i \to n + d - x_i$.
- Taking the complement of a sequence gives a one to one correspondance between the generators and relations of A_s^d and $(A_{d+2}^{s-2})^\dagger$.

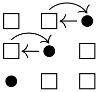

Figure – illustrating the duality of the relations

Results

Theorem (E)

The algebra of the poset $J(P_{m,n})$ is derived equivalent to the Higher Auslander algebra A^{n-1}_{m+1}.

Corrolary, well known ?

$A^{n-1}_{m+1} \cong (A^{m-1}_{n+1})^{\dagger}$.

Dedda 2023, Corollary 1.4

There is a quasi equivalence of triangulated A_∞-categories $\mathcal{F}(f_{s,d}) \xrightarrow{\cong} \mathbf{perf}(A^d_s)$between the derived Fukaya-Seidel category of a singularity $f_{s,d}$ and the perfect derived A_∞-category of the $\mathbb{k}$-algebra A^d_s.

Recall

Tilting object

An object T of $D^b(A)$ is tilting if it has no self extension and Thick $T = D^b(A.)$

Theorem (Rickard 1989, Keller (see König et Zimmermann 1998))

Let A and B be $\mathbb{k}$-algebras which are flat as modules over $\mathbb{k}$. The following are equivalent

- *There is a $\mathbb{k}$-linear triangulated equivalence* $(F, \phi) : D(A\text{-Mod}) \to D(B\text{-Mod})$ *;*
- *There is a tilting complex of B modules T such that* $(\mathrm{Hom}_{D^b(A)}(T, T))^{op} \cong A$*.*

Strategy for the proof of Theorem E

Theorem (3.3.4)

Let C be a *boolean* *antichain of a lattice L. Let $I \subseteq L$ be an interval. There exists at most one integer p such that $\mathrm{Hom}_{D^b}(M_C[0], I[p])$ is non zero. When such an integer exists, the hom space is one dimensional.*

$\rightsquigarrow$Extract a *suitable* tilting object from the Yıldırım intervals.

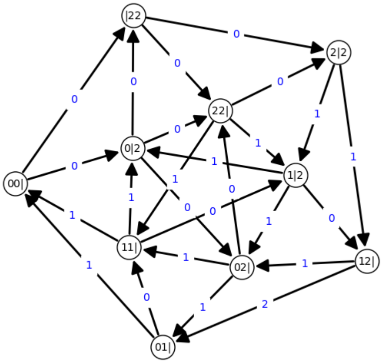

Figure – Graph of the subcategory of $D^b(J(P_{2,2}))$ spanned by the objects $\mathcal{P}_\alpha$ introduced byYıldırım

Decomposition

Proposition 5.1.12

Let $\phi : \mathcal{P}_\alpha \to \mathcal{P}_\beta[i]$ be a non zero morphism in $\mathcal{Y}_{m,n}$. Then there exists a unique subset J of S_α such that ϕ factors through $\mathcal{P}_{q_J(\alpha)}[i]$ and $|J| = i$ completing the following commutative diagram.

$$\begin{tikzcd}
\mathcal{P}_\alpha \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{P}_\beta[|J|] \\
& \mathcal{P}_{q_J(\alpha)}[|J|] \arrow[ur] &
\end{tikzcd}$$

The two maps that we thus obtain can be further decomposed into irreducible degree zero and irreducible degree one morphisms, denoted ${}^0u_k$ and ${}^1u_k$ for suitable k and l.

Definition

$C_{m,n}$ = increasing sequences of length m in $\{-m, \cdots -1, 0, 1, \ldots, n\}$.

Map

$$J_{m,n} \to C_{m,n}$$
$$(\lambda_1^{\mu_1}, \ldots, \lambda_r^{\mu_r}) \mapsto \{\hat{x_r}, \ldots, \hat{x_1}, \ldots, \lambda_1, \ldots, \lambda_r\}$$

Where $x_i = \sum_{j=1}^{i} \mu_j \rightsquigarrow$ ending index of the value λ_i in the partition

This map is injective. There is a bijection between configurations and *enhanced partitions* Yldrm 2019.

Different presentations for one category

Corollary 5.2.14

The morphisms in $\mathcal{Y}_{m,n}$ are generated by

1. the maps u_k^R, with relations generated by $\rho_{k,l}^R = u_l^{\sigma_k^-(R)} \circ u_k^R - \varepsilon u_k^{\sigma_l^-(R)} \circ u_l^R$ with $\varepsilon = -1$ if k and l are positive and 1 otherwise, along with $z_k^R = u_k^{\sigma_{k-1}^-(R)} \circ u_k^R$;
2. the maps $v_k^R = (-1)^{\kappa(R,k)} \cdot u_k^R$, with relations generated by $(\rho'_{k,l})^R = v_l^{\sigma_k^-(R)} \circ v_k^R - v_k^{\sigma_l^-(R)} \circ v_l^R$ along with $(z'_k)^R = v_{k-1}^{\sigma_k^-(R)} \circ v_k^R$;
3. the morphisms $w_k^R = \varepsilon(R,k)(u_k^R)$ with $\varepsilon(R,k) = (-1)^{\nu(R,k)+\kappa_R}$ if $k < 0$ and $\varepsilon(R,k) = 1$ otherwise, with relations generated by $(\rho''_{k,l})^R = w_l^{\sigma_k^-(R)} \circ w_k^R + w_k^{\sigma_l^-(R)} \circ w_l^R$ along with $(z''_k)^R = w_{k-1}^{\sigma_k^-(R)} \circ w_k^R$.

Take away from the previous slide

We are able to modulate signs ! Informally, in this very specific case, we managed to do the following :

$$ab - cd = 0 \leftrightarrow a'b' + c'd' = 0$$

Key : the quantities $\kappa(R, k)$ and $\nu(R, k)$.

Extracting a tilting complex

Set

$$T := \bigoplus_{\alpha \in J_{m,n}} \mathcal{P}_\alpha[\kappa_\alpha]. \tag{2}$$

This is a tilting complexe. Using the presentations 2 and 3 of the category $\mathcal{Y}_{m,n}$ we get

$$\mathrm{End}(T) \cong A_{m+1}^{n-1} \cong (A_{n+1}^{m-1})^\dagger.$$

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Different presentations for one category I

Here is the correct and complete formulation of Corollary 5.2.14. The morphisms in $\mathcal{Y}_{m,n}$ are generated by

1. the maps u_k^R for all $R \in C_{m,n}$ and for all $k \in R$ allowed, with relations generated by $\rho_{k,l}^R = u_l^{\sigma_k^-(R)} \circ u_k^R - \varepsilon u_k^{\sigma_l^-(R)} \circ u_l^R$ where $k, l \in R$ such that $k-1, l-1 \notin R$ and $\varepsilon = -1$ if k and l are positive and 1 otherwise, along with $z_k^R = u_k^{\sigma_{k-1}^-(R)} \circ u_k^R$ where $k \in R$, but $k-1, k-2 \notin R$;
2. the maps $v_k^R = (-1)^{\kappa(R,k)} \cdot u_k^R$ for all $R \in C_{m,n}$ and for all $k \in R$ allowed, with relations generated by $(\rho'_{k,l})^R = v_l^{\sigma_k^-(R)} \circ v_k^R - v_k^{\sigma_l^-(R)} \circ v_l^R$ where $k, l \in R$ such that $k-1, l-1 \notin R$ along with $(z'_k)^R = v_{k-1}^{\sigma_k^-(R)} \circ v_k^R$ where $k \in R$, but $k-1, k-2 \notin R$;

Different presentations for one category II

3. the morphisms $w_k^R = \varepsilon(R,k)(u_k^R)$ for all $R \in C_{m,n}$ with $\varepsilon(R,k) = (-1)^{\nu(R,k)+\kappa_R}$ if $k < 0$ and $\varepsilon(R,k) = 1$ otherwise, for all $k \in R$ allowed, with relations generated by $(\rho''_{k,l})^R = w_l^{\sigma_k^-(R)} \circ w_k^R + w_k^{\sigma_l^-(R)} \circ w_l^R$ where $k, l \in R$ such that $k-1, l-1 \notin R$ along with $(z''_k)^R = w_{k-1}^{\sigma_k^-(R)} \circ w_k^R$ where $k \in R$, but $k-1, k-2 \notin R$.

A poset which is not fractionally Calabi–Yau

From Diveris, Purin et Webb 2017

$$\begin{tikzcd}
 & \circ \arrow[dl] \arrow[ddr] & \\
\circ \arrow[d] & & \\
\circ \arrow[d] \arrow[dr] & & \circ \arrow[dl] \arrow[d] \\
\circ \arrow[dr] & \circ \arrow[d] & \circ \arrow[dl] \\
 & \circ &
\end{tikzcd}$$