Exercise sheet 6
Abelian Categories

February 12, 2026

The following exercises will be discussed in class.

Exercise 1

Let $\mathcal{C}$ be an **Ab**-category. Let X, Y be two objects of $\mathcal{C}$.

(i) Show that the product of X and Y exists *if and only if* their coproduct exists. Show moreover that in that case the product and the coproduct are isomorphic. We call the resulting object the biproduct.

(ii) Show that the biproduct of X and Y is characterized by an object Z together with morphisms

$$X \underset{\pi_X}{\overset{\iota_X}{\rightleftarrows}} Z \underset{\iota_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$$

that satisfy the following equalities

$$\pi_X \iota_X = id_X, \quad \pi_Y \iota_Y = id_Y, \quad \iota_X \pi_X + \iota_X \pi_X = id_Z.$$

Exercise 2

Let $\mathcal{C}$ be a pre-abelian category. Let $\phi : X \to Y$ be a morphism of $\mathcal{C}$.

(i) Show that there exists a canonical morphism $\overline{\phi} : \operatorname{coker} \ker \phi \to \ker \operatorname{coker} \phi$ such that ϕ decomposes as

$$X \xrightarrow{\pi} \operatorname{coker} \ker \phi \xrightarrow{\overline{\phi}} \ker \operatorname{coker} \phi \xrightarrow{\iota} Y$$

where π is an epimorphism and ι a monomorphism.

(ii) Show that $\overline{\phi}$ is an isomorphism for every ϕ if and only if $\mathcal{C}$ is an abelian category.

Homework The following exercises will not be discussed in class and can be handed in until 26/01/2026. Feel free to only hand in partial answers or to ask for more details by email.

Exercise 1

Let $\mathcal{C}$ be an abelian category and let P be an object in $\mathcal{C}$. Show that the following statements are equivalent.

(a) P is a projective object.

(b) $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is an exact functor.

(c) Every short exact sequence $0 \to A \to B \to P \to 0$ splits.

Exercise 2

Let R be a ring, not necesserally commutative. We consider the abelian category of left R-modules and we denote it R-Mod. Let P be a left R-module. Show that the following statements are equivalent.

(a) P is a projective module.

(b) There exists a set I and an R-module Q such that $P \oplus Q \cong R^{(I)}$. Here $R^{(I)}$ denotes the free R-module over the set I.

(c) There exists a set I, a collection $\{x_i\}_{i \in I}$ of elements of P and a collection $\{f_i\}_{i \in I}$ of element of $\mathrm{Hom}_R(P, R)$ such that for all $x \in P$

- there exist finitely many $i \in I$ with $f_i(x) \neq 0$;
- $x = \sum_{i \in I} f_i(x) x_i$.