Winter Semester 2025
Ruhr Universität Bochum

Dr. **Azzurra** Ciliberti, Dr. **Tal** Gottesman
Category Theory

Exercise sheet 5
Adjunctions and equivalences

December 11, 2025

The following exercises will be discussed in class.

Exercise 1

Prove that a functor $F : \mathcal{C} \to \mathcal{D}$ establishes an equivalence of categories between locally small categories[1] $\mathcal{C}$ and $\mathcal{D}$ if and only if F is fully faithfull and essentially surjective.

Hint: You can use, with or without proof, the following result.

Lemma 1. *Let $A \xrightarrow{\phi} A'$ and $B \xrightarrow{\psi} B'$ be isomorphisms in a category $\mathcal{C}$. Let $f : A \to B$ be a morphism in $\mathcal{C}$. Then there is a unique morphism f' which makes the following four diagrams commute*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[\psi]{} & B' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow[\psi^{-1}]{} & B' \end{array} \qquad \begin{array}{ccc} A & \xleftarrow{\phi^{-1}} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[\psi]{} & B' \end{array} \qquad \begin{array}{ccc} A & \xleftarrow{\phi^{-1}} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow[\psi^{-1}]{} & B' \end{array}$$

Exercise 2

Prove that if functors F and G establish an equivalence between categories $\mathcal{C}$ and $\mathcal{D}$, then they form an adjoint pair. You can proceed by replacing one of the originally specified natural isomorphisms by a new well chosen unit or counit.

[1] This assumption is added for convenience.

Homework The following exercises will not be discussed in class and can be handed in until 7/01/2026. They are somewhat longer then exercises from previous weeks. Feel free to only hand in partial answers or to ask for more details by email.

Exercise 1

Fix a field $\Bbbk$. Let P be a finite poset. We call a functor $P \to Vect_{\Bbbk}^{fd}$ a finite dimensional representation of the poset P. We denote by $\mathrm{Rep}(P, \Bbbk)$ the cateogry whose objects are the representations of P and the morphisms are natural transformations between these representations. The incidence algebra $\mathcal{I}(P)$ of P is the finite dimensional $\Bbbk$-algebra whose basis as a $\Bbbk$ vector sapce is formed of the relations $(x \leq y)$ between elements in P and whose multiplication is defined as follows:

$$(x \leq y) \cdot (z, t) = \begin{cases} (x, t) & \text{if} y = z \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Prove that the category of left modules over $\mathcal{I}(P)$ is equivalent to the category $\mathrm{Rep}(P, \Bbbk)$.

Exercise 2

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $K : I \to \mathcal{C}$ a diagram.

- We say that F*preserves* the limit of K if when the limit L of K exists in $\mathcal{C}$ and $F(L)$ is the limit of the diagram FK in $\mathcal{D}$;
- We say that F *reflects* the limit of K if when there exists an object L in $\mathcal{C}$ such that $F(L)$ is the limit of FK in then L is the limit of K in $\mathcal{C}$.
- We say that *creates* limit of K if when the limit of FK exists in $\mathcal{D}$ there exists L in $\mathcal{C}$ which is the limit of K and whose image under F is the limit of $\mathcal{D}$.

(i) Give the dual definitions for colimits

(ii) Prove that an equivalence of categories preserves reflects and creates all limits and colimits.

Exercise 3

Suppose $F : \mathcal{C} \to Set$ is equivalent to $G : \mathcal{D} \to Set$ in the sense that there is an equivalence of categories $H : \mathcal{C} \to \mathcal{D}$ so that GH and F are naturally isomorphic.

(i) If G is representable, then is F representable?

(ii) If F is representable, then is G representable?

Exercise 4

Use the General Adjoint Functor Theorem to prove that the inclusion $Haus \to Top$ of the full subcategory of Hausdorff spaces into the category of all spaces has a left adjoint. The left adjoint carries a space to its “largest Hausdorff quotient.”