

Abelian categories II : From split sequences to projective modules

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In the previous notes, we defined *abelian categories*. *Short exact sequences* played a role in defining *exact functors*, the functors that preserve the abelian structure. In these notes we expand on the notion of short exact sequences and its connection to important functors.

1 Split Exact sequences

Throughout these notes let \mathcal{C} be an abelian category. Let A and C be two objects in \mathcal{C} . Recall that abelian categories are complete and cocomplete. Exercise 2. of exercise sheet 6, shows that for any Ab-category, and in particular for an abelian category, the colimit of the diagrams over discrete categories coincide. Let e denote both the limit and the colimit of the diagram D over the discrete category $I := \{1, 2\}$.

$$\begin{aligned} D : I &\rightarrow \mathcal{C} \\ 1 &\mapsto A \\ 2 &\mapsto C \end{aligned}$$

We denote by ι_A, ι_C, π_A and π_C the canonical maps defining the corresponding universal cones and cocones. In the exercise we show that the following equalities hold.

$$\pi_A \iota_A = id_A, \quad \pi_C \iota_C = id_C, \quad \iota_A \pi_A + \iota_C \pi_C = id_{A \oplus C}$$

The next Lemma connects coproducts to short exact sequences with special properties.

Lemma 1 (splitting Lemma). *Let $E : 0 \rightarrow A \xrightarrow{q} B \xrightarrow{r} C \rightarrow 0$ be a short exact sequence in \mathcal{C} . Then the following assertions are equivalent.*

(a) *There exists a morphism $t : B \rightarrow A$ such that $tq = id_A$. We say that E is left split.*

(b) *There exists a morphism $u : C \rightarrow B$ such that $ru = id_C$. We say that E is right split.*

(c) There exists an isomorphism $h : B \rightarrow A \oplus C$ such that $hq = \iota_A : A \hookrightarrow B$ and $rh^{-1} = \pi_C : B \twoheadrightarrow C$.

Proof. ((a) \Rightarrow (b)) Consider the morphism $id_B - qt$. Because $(id_B - qt)q = 0$ and r is the cokernel of q , there exists a unique map $u : C \rightarrow B$ such that $ur = id_B - qt$. Then the following computation

$$(ru - id_C)r = rur - r = r - rqt - r = r - r = 0$$

shows that $ru = id_C$.

((b) \Rightarrow (a)) Dually, consider the map $id_B - ur$ to construct the required $t : B \rightarrow A$.

((a) + (b) \Rightarrow (c)) Items (a) and (b) give us maps that define a cone (resp. cocone) over the objects A and C with summit (nadir) B . We start by showing that the cone and the cocone are universal. The uniqueness of these morphisms will be discussed in the exercise class. It is then possible to construct the isomorphism h using the universal property of B and $A \oplus C$.

((c) \Rightarrow (a), (b)) Set $u = h^{-1}\iota_C$ and $t = \iota_A h$. The remaining details are left to the reader. □

If E satisfies conditions (a), (b) and (c) from the previous lemma, we say that E is *split*, t is a *retraction* and u is a *section*.

2 Hom functors and exactness

In the first weeks of this course, we saw the importance of Hom functors are ubiquitous in category theory, from Yoneda's Lemma to the different notions of universal properties. In this section, show how Hom functors act on short exact sequences in abelian categories, *i.e.* the structure of the abelian category.

Lemma 2. *Let X be an object in \mathcal{C} . Then*

(a) *the functor $\text{Hom}_{\mathcal{C}}(X, -)$ is (covariant), additive and left exact;*

(b) *the functor $\text{Hom}_{\mathcal{C}}(-, X)$ is (contravariant), additive and right exact.*

The statement about additivity holds for any linear category. This part is left as an exercise.

Proof. We prove statement (a). The proof of statement (b) is dual. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$ be a short exact sequence in \mathcal{C} . We want to argue that

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(X, A) \xrightarrow{f^*} \text{Hom}_{\mathcal{C}}(X, B) \xrightarrow{g^*} \text{Hom}_{\mathcal{C}}(X, C)$$

is a short exact sequence of abelian groups. In this category, morphisms are monic if and only if they are injective and epic if and only if they are surjective.

- exactness at $\text{Hom}_{\mathcal{C}}(X, A)$: the morphism f is monic. It follows that for any $\phi, \psi \in \text{Hom}_{\mathcal{C}}(X, A)$, if $f\phi = f\psi$ then $\phi = \psi$. Hence the morphism f_* is injective, *monic*.
- exactness at $\text{Hom}_{\mathcal{C}}(X, B)$: consider $\phi \in \ker(g_*) \subseteq \text{Hom}_{\mathcal{C}}(X, B)$ i.e. $g_*(\phi) = 0$. Then, by the universal property of the kernel, ϕ factors through $f = \ker(g)$ i.e. there exists a map $\psi \in \text{Hom}_{\mathcal{C}}(X, A)$ such that $\phi = f\psi = f_*(\psi) \in \text{im}(f_*)$.

□

Question 1. When are the functors $\text{Hom}_{\mathcal{C}}(X, -)$ or $\text{Hom}_{\mathcal{C}}(-, X)$ exact?

3 Projective and Injective objects

Definition 3 (Projective and injective objects). An object P of a \mathcal{C} is a *projective object* if for any epimorphism $\phi : M \rightarrow N$ and morphism $\psi : P \rightarrow N$, there exists a morphism $f : P \rightarrow M$ such that $\phi f = \psi$. This is summarised in the following commutative diagram.

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow \exists f & \downarrow \psi & & \\
 M & \xrightarrow{\phi} & N & \longrightarrow & 0
 \end{array}$$

Dually, an object I of a \mathcal{C} is an *injective object* if for any monomorphism $\phi : M \rightarrow N$ and morphism $\psi : M \rightarrow I$, there exists a morphism $f : N \rightarrow I$ such that $f\phi = \psi$. This is summarised in the following commutative diagram.

$$\begin{array}{ccccc}
 0 & \longrightarrow & M & \xrightarrow{\phi} & N \\
 & & \downarrow \psi & \swarrow \exists f & \\
 & & I & &
 \end{array}$$

Example 4. In the category of abelian groups \mathbf{Ab} , the projective modules are precisely the free modules. We illustrate the fact that groups with torsion cannot be projective as the following diagram cannot be completed.

$$\begin{array}{ccccc}
 & & \mathbb{Z}/2\mathbb{Z} & & \\
 & \swarrow \exists f & \downarrow \psi & & \\
 \mathbb{Z} & \xrightarrow{\phi} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
 \end{array}$$

We can now answer Question 1 as well as connect the notion of projective modules to split exact sequences.

Lemma 5. Let P be an object in the category \mathcal{C} . The following statements are equivalent.

- (a) P is a projective object.
- (b) The functor $\text{Hom}_{\mathcal{C}}(P, -)$ is exact.
- (c) Every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$, splits.

Proof. ((a) \Leftrightarrow (b)) Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence in \mathcal{C} . We want to show that

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(P, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{C}}(P, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{C}}(P, C) \rightarrow 0$$

is exact if and only if P is projective. By Lemma 2, we only need to discuss the surjectivity of the map g_* . It is then easy to see that g_* is surjective if and only if we can complete the following diagram

$$\begin{array}{ccc} & P & \\ \swarrow \exists & \downarrow \phi & \\ B & \xrightarrow{g} C & \longrightarrow 0 \end{array}$$

for any $\phi \in \text{Hom}_{\mathcal{C}}(P, C)$ which makes P projective.

((a) \Rightarrow (c)) Let $0 \rightarrow A \rightarrow B \xrightarrow{r} P \rightarrow 0$ be a short exact sequence in \mathcal{C} . Then consider the diagram below

$$\begin{array}{ccccccc} & & & P & & & \\ & & & \downarrow id_P & & & \\ & & \swarrow \exists u & & \searrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{r} & P \longrightarrow 0 \end{array}$$

to construct a section u of the map r , showing that the sequence splits. (Lemma 1).

((c) \Rightarrow (a)) For this last item we use the following fact without proof: let $\phi : X \rightarrow Y$ be an epimorphism. Then a pullback diagram of the form

$$\begin{array}{ccc} T & \xrightarrow{g} & Z \\ \downarrow \iota & \ulcorner & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

is also a pushforward and ψ is also an epimorphism. We apply this as follows. Let $\phi : M \rightarrow N$ be an epimorphism and let $\psi : P \rightarrow N$ be a morphism in \mathcal{C} . We construct the pull back over these two maps. See the diagram below.

$$\begin{array}{ccccccc} 0 & & \text{ker} & \longrightarrow & P' & \xrightarrow{\exists u} & P \\ & & & & \downarrow \iota & \ulcorner & \downarrow \psi \\ & & & & M & \xrightarrow{\phi} & N \longrightarrow 0 \end{array}$$

By assumption, the top exact sequence splits. We use its section to construct the dotted arrow above, showing that P is projective. □

4 Projective modules in abelian categories

Projective objects in modules categories have several other convenient descriptions. In the following Lemma we denote by R^I the free R -module over a set I . We denote the standard generators of $R^{(I)}$ by e_i for each $i \in I$. Elements of R^I are finite linear combinations of these generators. This is the infinite coproduct of I copies. It comes with canonical projections on each component $\pi_i : R^{(I)} \rightarrow R_i$, where R_i denotes the the i^{th} copie of the ring R .

Lemma 6. *Let P be a module over a ring R . Then the following statements are equivalent.*

- (a) P is a projective object in the abelian category $R\text{-Mod}$.
- (b) There exists a set I and an R -module Q such that $P \oplus Q \cong R^{(I)}$.
- (c) There exists a sets I , $\{a_i \in P\}$ and $\{f_i \in \text{Hom}_{R\text{-Mod}}(P, R)\}$ such that for all element $x \in P$ there exists finitely many elements i of I such that $f_i(x) \neq 0$ and $x = \sum_I f_i(x)a_i$.

Proof. ((a) \Rightarrow (b)) Consider the epimorphism

$$\begin{aligned} \phi : R^{(P)} &\rightarrow P \\ e_x &\rightarrow x \end{aligned}$$

As P is projective, by item (c) of Lemma 5 this map has a section. We conclude by applying item (a) of Lemma 1. Note: we are implicitly considering the short exact sequence

$$0 \rightarrow \ker \phi \rightarrow R^{(P)} \xrightarrow{\phi} P \rightarrow 0.$$

((b) \Rightarrow (a)) Suppose P is a direct summand of $R^{(I)}$. Denote π and ι the corresponding canonical maps associated to P in that coproduct. Let $r : B \rightarrow P$ be an epimorphism. Because the module $R^{(I)}$ is free and in particular projective. Hence the following diagram can be completed by a map ψ .

$$\begin{array}{ccc} & R^{(I)} & \\ & \swarrow \exists \psi & \downarrow \pi \\ B & \xrightarrow{\phi} & P \longrightarrow 0 \end{array}$$

Then the map $s = \psi \iota$ is a section of r . We have recovered item (c) of Lemma 5.

((b) \Rightarrow (c)) By assumption we have the following maps, given by coproduct structures.

$$\begin{array}{ccccc}
 & \xrightarrow{\iota_i} & & \xrightarrow{\phi} & \\
 R_i & & R^{(I)} & & P \\
 & \xleftarrow{\pi_i} & & \xleftarrow{\iota} & \\
 & & & &
 \end{array}$$

Set $a_i = \pi(e_i)$ and $f_i = \pi\iota$. We can then check that

$$\begin{aligned}
 x = \pi\iota(x) &= \pi\left(\sum_{i \in I} \iota_i \pi_i\right)\iota(x) = \pi\left(\sum_{i \in I} \iota_i f_i\right)(x) = \pi\left(\sum_{i \in I} f_i(-)e_i\right)(x) \\
 &= \pi\left(\sum_{i \in I} f_i(x)e_i\right) = \sum_{i \in I} f_i(x)a_i
 \end{aligned}$$

The elements of $R^{(I)}$ are finite linear combinations of the standard generators. Considering the element $\sum_{i \in I} f_i(x)e_i$ of $R^{(I)}$, we can conclude that there exists finitely many $i \in I$ such that $f_i(x) \neq 0$.

((c) \Rightarrow (b)) Consider the epimorphism

$$\begin{aligned}
 \phi : R^{(I)} &\rightarrow P \\
 e_i &\mapsto a_i
 \end{aligned}$$

Then one can check that the following map is a section of ϕ

$$\begin{aligned}
 P &\rightarrow R^{(I)} \\
 x &\mapsto \sum_{i \in I} f_i(x)e_i.
 \end{aligned}$$

We then again conclude using item (a) of Lemma 1. □

Remark 7. There is a dual version of Example 4 and Lemma 5 for injective modules. To my knowledge there is no dual version of Lemma 6. This is why some representation theories tend to prefer to work with projective modules and not injective modules, even though from a categorical perspective that are dual to one another.

To conclude these notes we point out that one of the goals of the field of *Homological Algebra* is precisely to study the failure of exactness of the Hom functors and their left adjoints the tensor products. Lemma 2, Lemma 5 and Lemma 6 are the first steps in this study. **In the next session** we study more properties of projective object in modules categories where we will use the abstract language of categories together with specific properties of modules to reach satisfying results. In particular, we will give more information about the tensor product and prove Morita's theorem which characterises equivalences between module categories.

References

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