Interractions between limits and colimits

Tal Gottesman

December 27, 2025

1 Summary of the previous week

Fix $\mathcal{C}$ a category and I a small category. Let $\mathcal{C}^I$ denote the category of functors from I to C. For a morphism α in the category I we denote by $\mathrm{dom}(\alpha)$ its domain and by $\mathrm{codom}(\alpha)$ its codomain. We call a functor $F : I \to \mathcal{C}$ a diagram. In previous lectures we introduced the functors of $\mathrm{Cone}(-, F) : \mathcal{C} \to Set$ and $\mathrm{Cocone}(F, -) : \mathcal{C} \to Set$. The first one is contravariant and the second is covariant. We also defined the limit $\lim_I F$ of the diagram F to be, when it exists, in one of the following equivalent ways:

1. the representative of the functor $\mathrm{Cone}(-, F)$ along with a universal element in $\mathrm{Cone}(\lim_I F, F)$;
2. A terminal object in the category of elements of the functor $\mathrm{Cone}(-, F)$. Concretely, this category has cones for over F for objects and maps that commute with the *legs* of the cones for morphisms.

Dually, the colimit $\mathrm{colim}_I F$ of the diagram F when it exists is

1. The representative of the functor $\mathrm{Cone}(F, -)$ with a universal element in $\mathrm{Cocone}(F, \lim_I F)$;
2. An Initial object in the category of elements of the functor $\mathrm{Cone}(F, -)$. Again, this category has ocones for under F for objects and maps that commute with the *legs* of the cocones for morphisms.

In other words, when they exist, limits and colimits are a special case of universal properties for contravariant or covariant functors respectively.

Question 1. *What is your favourite point of view to describing limits and colimits? Is it one of the above or do you have another source of intuition?*

Definition 1. We say that a category is complete (resp. cocomplete) if admits all small limits (resp. colimits)

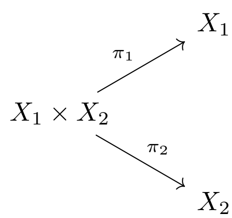
In the previous lecture we showed that the category of sets admits all small limits. We illustrate this result with a concrete example, giving as many details as we can.

2 Products and coproducts of sets

Products, (respectively co-products) are the limits (resp. colimits) over diagramss of shape

$$I = \bullet \qquad \bullet$$

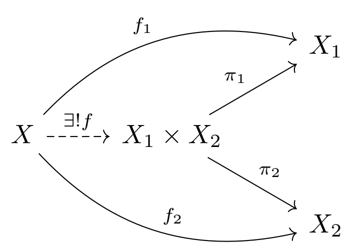
In the category of sets, they correspond to cartesian products (resp. disjoint unions) of sets. Let X_1 and X_2 be two sets. The diagram below shows how $X_1 \times X_2$ fits in a cone over a diagram $F : I \to Set$ which sends one object of I to X_1 and the other to X_2.



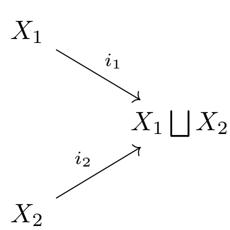
Here the maps π_1 and π_2 are the familiar projection maps. The fact that $X_1 \times X_2$ along with the legs of this cone is the limit over the diagram F follows from the observation that any tuple of morphisms with the same source X and targets X_1 and X_2 define in a unique way a morphism from X to $X_1 \times X_2$ as follows.

$$\begin{aligned} \text{Cone}(X, F) \cong Set(X, X_1) \times Set(X, X_2) &\xrightarrow{\sim} Set(X, X_1 \times X_2) \\ (f_1, f_2) &\mapsto (f : x \mapsto (f_1(x), f_2(x))). \end{aligned}$$

This isomorphism illustrates the first point of view for the limit. To see the limit as a terminal object in the category of cones over F it is customary to rewrite the above bijection as a diagram as follows.



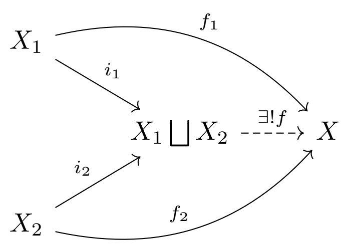
Similarly, $X_1 \bigsqcup X_2$ fits in a natural cocone under the same diagram where the legs of the cocone are the canonical inclusions.



Then the colimit nature of $X_1 \bigsqcup X_2$ is either expressed by the isormorphism of functors

$$\begin{aligned} \text{Cocone}(F, X) \cong Set(X_1, X) \times Set(X_2, X) &\xrightarrow{\sim} Set(X_2 \bigsqcup X_2, X) \\ (f_1, f_2) &\mapsto (f : x \mapsto f_i(x) \text{ if } x \in X_i). \end{aligned}$$

or by the following diagram.



Remark 2. In the category of vector spaces over a field $\Bbbk$, products and coproducts coincide, forming so called "biproducts". This will be discussed later when we encounter the notion of categories *enriched* in other categories.

Definition 3. A limit of a diagram of shape $\bullet \rightrightarrows \bullet$ is called an *equalise*r. Dually, a colimit of a diagram of the same shame is called a *coequaliser*.

The above constructions of products and coproducts extend to products and coproducs over a small category I whose set of objects has arbitrary cardinal.

Theorem 4. *The category of sets is complete and cocomplete. Moreover, limits (resp. colimits) can be computed using only products and equalisers (resp. coproducts and coequalisers). More precisely, if $F : I \to Set$ is a diagram over a small category I, then*

- *the limit of F exist and is the equaliser of the following diagram*

$$\Pi_{i \in I} F(i) \underset{c}{\overset{d}{\rightrightarrows}} \Pi_{\alpha \in I} F(\text{codom}(\alpha)).$$

In this equation, α denotes a morphism in the category I, the map c is defined by $c((a_i)_i \in I) = (a_{\text{codom}(\alpha)})_{\alpha \in I}$ and the map d is defined by $d((a_i)_i \in I) = (F(\alpha)(a_{\text{dom}(\alpha)}))_{\alpha \in I}$.

- *the colimit of F exist and is the coequaliser of the following diagram*

$$\Pi_{\alpha \in I} F(\text{dom}(\alpha)) \underset{c}{\overset{d}{\rightrightarrows}} \Pi_{i \in I} F(i).$$

In this equation, α denotes a morphism in the category I, the map c is defined by $c(a) = F(\alpha)(a)$ for $a \in \text{dom}(\alpha)$ and the map d is defined by $d(\alpha) = \alpha$.

For a proof of this result we direct the reader to theorem 3.2.13 and 3.4.12 of the book *Categories in Context* by Emily Riehl that we will refer to as *the book* in the rest of these notes.

3 Limits and functors

In this section we collect two useful results about functors that are either well behaved with regards to limits or that are defined using limits. Both results admit dual version with colimits.

Definition 5. We say that a covariant functor $G : \mathcal{C} \to \mathcal{D}$ *preserves* limits if it sends the limit of all diagrams $F : I \to \mathcal{C}$ to the limit of the diagram $GF : I \to \mathcal{D}$ whenever the limit of F exists in $\mathcal{C}$.

Theorem 6. *As before, let I be a small category and let $\mathcal{C}$ be a locally small category. Suppose that the limit of the diagram $F : I \to \mathcal{C}$ exists. Then we have the following isomorphism*

$$\mathcal{C}(X, \lim_I F) \cong \lim_I \mathcal{C}(X, F(-)).$$

In other words, $\mathcal{C}(X, -)$ preserves limits

This theorem gives us the opportunity to point how the limits in the category of sets play an important role for limits in locally small categories in general. See Theorem 3.4.7 of the book for the dual of this theorem.

Proof. We want to show that $\mathcal{C}(X, \lim_I F)$ is the limite of the I shaped diagram $\mathcal{C}(X, F(-))$. We choose to do so by constructing a universal cone over $\mathcal{C}(X, F(-))$ with summit $\mathcal{C}(X, \lim_I F)$. Denote $\lambda_i : \lim_I F \to F(i)$ the legs of the universal cone over the diagram F. For each $i \in I$ consider the maps

$$\begin{aligned}(\lambda_i)_\star : \mathcal{C}(X, \lim_I F) &\to \mathcal{C}(C, F(i)) \\ f &\mapsto \lambda_i \circ f.\end{aligned}$$

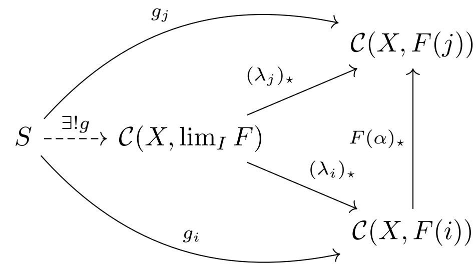
Recall that the arrows of the diagram $\mathcal{C}(X, F(-))$ are precisely the post composition maps $F(\alpha)_\star$ for each arrow α in the category I. Then we have

$$F(\alpha)_\star \circ (\lambda_{\mathrm{dom}(\alpha)})_\star = (F(\alpha) \circ \lambda_{\mathrm{dom}(\alpha)})_\star = (\lambda_{\mathrm{codom}(\alpha)})_\star.$$

Suppose we have cone with summit S over $\mathcal{C}(X, F(-))$ with legs

$$g_i : S \to \mathcal{C}(X, F(i)).$$

Note that for each element s of S, the collection of maps $(g_i(s) : X \to F(i))$ defines a cone with summit X over the diagram F. Using the universal property of the limite of F we get a unique map $g(s) : X \to \lim_I F$. We have thus defined a map $g : S \to \mathcal{C}(X, \lim_I F)$ which fits in the following familly of commutative diagrams.



It remains to argue that g is unique. This follows from the universal property of $\lim_I F$

□

Proposition 1. *Suppose that $\mathcal{C}$ admits all J shapped limits. Then any choice of a limit for each diagram can be extended into a functor* $\lim_I : \mathcal{C}^I \to \mathcal{C}$

Proof. We want to construct a functor as follows

$$\begin{aligned} \mathcal{C}^I &\to \mathcal{C} \\ F &\mapsto \lim_I F \\ (\alpha : F \to G) &\mapsto (\lim_I F \to \lim_I G) \end{aligned}$$

where $\lim_I F$ is the limit we have arbitrarily chosen for each diagram F. All we need to do is explain what the morphism f_α is, given a natural transformation $\alpha : F \to G$ between two diagrams over I. Denote by λ_i the legs of the universal cone over F with summit $\lim_I F$. Because α is a natural transformation, we can construct a cone with summit $\lim_I F$ over G whose legs are $\alpha(i) \circ \lambda_i$ for each object i of I. By the universal property of $\lim_I G$ we obtain a unique map $f_\alpha : \lim_I F \to \lim_I G$ that commutes with the legs of the cones. It remains to check that this assignment is compatible with composition of morphisms *i.e.* that given two natural transformations $\alpha : F \to G$ and $\beta : G \to H$, we have

$$f_{\beta \circ \alpha} = f_\beta \circ f_\alpha.$$

One can check that both these morphisms fit in a diagram whose legs are $\beta(i) \circ \alpha(i) \circ \lambda_i$. They are thus equal by the universal property of $\lim_I H$. □

As an exercise, turn the proof above into a sequence of diagrams. Note also that it is quite rare that arbitrary choices can be made functorial. We conclude this section by pointing out that limits and colimits are always unique up to unique ismorphism. See Exercise 1 of the third exercise sheet.

4 Double Limits

Let I and J be small categories. We define the product of categories $I \times J$ to be the small category whose

- object set is $Obj(I \times J) = Obj(I) \times Obj(J)$.
- morphisms are $I \times J((i,j),(i',j')) = I(i,i') \times J(j,j')$.

Remark 7. In the construction of the product $I \times J$ we used products of the category of sets. the resulting construction is a product in the category of categories Cat. In fact one can show that both Cat and CAT are complete and cocomplete. See section Proposition 3.5.6 of the book.

Consider a diagram $F : I \times J \to \mathcal{C}$. Suppose that for all object i of the category I, the limit of the diagram $F(i,-) : J \to \mathcal{C}$ exits. Using the universal properies of each of these limits, we get a diagram $\lim_J F(-,j) : I \to \mathcal{C}$ whose limit, if it exists, we denote $\lim_{i\in I} \lim_{j\in J} F(i,j)$. Under the corresponding necessary assumptions we can also construct $\lim_{j\in J} \lim_{i\in I} F(i,j)$.

Theorem 8. *If* $\lim_{i\in I} \lim_{j\in J} F(i,j)$ *and* $\lim_{j\in J} \lim_{i\in I} F(i,j)$ *exist, then they are ismorphic and define the limit* $\lim_{I\times J} F$.

Proof. First, by the Yoneda Lemma, the claim is equivalent to the natural isomorphisms

$$\mathcal{C}(X, \lim_{i\in I} \lim_{j\in J} F(i,j)) \cong \mathcal{C}(X, \lim_{j\in J} \lim_{i\in I} F(i,j)) \cong \mathcal{C}(X, \lim_{I\times J} F).$$

Next by Proposition 6 we have the following isomorphisms of sets

$$\mathcal{C}(X, \lim_{i\in I} \lim_{j\in J} F(i,j)) \cong \lim_{i\in I} \mathcal{C}(X, \lim_{j\in J} F(i,j)) \cong \lim_{i\in I} \lim_{j\in J} \mathcal{C}(X, F(i,j)).$$

The previous observation is symmetric in I and J. Hence, because $\mathcal{C}$ is locally small, it suffices to prove the current result in the category of set which we do next ☐

Remark 9. The Yoneda Lemma requires the isomorphisms to be functorial in the variable X. The isomorphism in the second step is functorial as a concequence of it being the unique isomorphism between representatives of universal properties. A similar arguement holds for the isomorphisms in the next step.

Theorem 10. *Consider a diagram* $F : I \times J \to Set$ *such that*

$$\lim_{i\in I} \lim_{j\in J} F(i,j) \text{ and } \lim_{j\in J} \lim_{i\in I} F(i,j)$$

exist. Then they are isomorphic and define the limit $\lim_{I\times J} F$.

Proof. It suffices to prove that $\lim_{i\in I} \lim_{j\in J} F(i,j)$ is the limit of the diagram F the case of $\lim_{j\in J} \lim_{i\in I} F(i,j)$ being symmetric. Let $(f_i)_{\in I}$ be a cone over the I-indexed diagram whose vertices are the $\lim_{j\in J} F(i,j)$ with summit X. Then for each object i of I, we get a cone with summit X, denoted $(f_{i,j})_{j\in J}$ over the J-indexed diagram whose object are the $F(i,j)$. A computation shows that the collection of maps $(f_{i,j})_{(i,j)\in I\times J}$ defines a cone over the category $I \times J$ with summit X. Any morphism $\alpha : (i,j) \to (i',j') \in I \times J$ is a tuple (f,g) where

$f : i \to i'$ is a morphism of the category I and $g : j \to i'$ of the category J. Then the following equality hold

$$\begin{aligned} F(\alpha) \circ f_{i,j} = F(f,g) \circ f_{i,j} &= F(f, id_{j'}) \circ F(id_i, g) \circ f_{i,j} \\ &= F(f, id_{j'}) f_{i,j'} = f_{i',j'}. \end{aligned}$$

Conversely, any cone $(f_{i,j})_{i,j}$ over F restricts to cones over the J-shaped diagrams we get by fixing an element $i \in I$. By the universal property of the J-shaped limit we get maps f_i which form a cone over the I-shaped diagram whose vertices are the $\lim_{j\in J} F(i,j)$ We have shown that $\lim_{i\in I} \lim_{j\in J} F(i,j)$ and $\lim_{j\in J} \lim_{i\in I} F(i,j)$ satisfy the same universal property. They are thus isomorphic and this concludes the proof ☐

Dually, colimits also commute with colimits.

5 Putting limits and colimits together

The main take-away of this section is that limits and colimits do not commute in general but can be *compared* with a canonical morphism.

Lemma 11. *Let $F : I \times J \to \mathcal{C}$ be a bifunctor. Suppose that $\lim_{j\in J} F(-,j)$, $\mathrm{colim}_{i\in I} \lim_{j\in J} F(i,j)$, $\mathrm{colim}_{i\in I} F(-,j)$ and $\lim_{j\in J} \mathrm{colim}_{i\in I} F(i,j)$ exist. Then there is a canonical map*

$$\kappa : \operatorname*{colim}_{i\in I} \lim_{j\in J} F(i,j) \to \lim_{j\in J} \operatorname*{colim}_{i\in I} F(i,j).$$

Proof. Consider the composition of maps indexed by objects in $(i,j) \in I \times J$

$$\kappa_{i,j} : \lim_{j'\in J} F(i,j) \xrightarrow{\pi_{i,j}} F(i,j) \xrightarrow{i_{i,j}} \operatorname*{colim}_{i'\in I} F(i',j).$$

By the magic of the colimits, *i.e* for fixed i these morphisms define a cone of shape J, this lifts to a morphism

$$\kappa_i : \lim_{j'\in J} F(i,j) \xrightarrow{\pi_{i,j}} F(i,j) \xrightarrow{i_{i,j}} \lim_{j\in J} \operatorname*{colim}_{i'\in I} F(i',j).$$

In turn, by the magic of the limits, *i.e* these morphisms define a cocone of shape I this lifts to the desired morphism κ. ☐

We did not go throught he details of the computations in this proof. They consist and showing that the legs of the cones and cocones we are considering indeed commute with the arrows of the diagrams. They do so because the maps $\pi_{i,j}$ and $i_{i,j}$ are themselves legs of a cone and a cocone and because the legs of the diagrams whose objects are the partial limits and colimits are themselves obtained by the universal properties and commute by construction with the correct $\pi_{i,j}$ and $i_{i,j}$. We encourage the reader to fill in the details.

Example 12 (limits of sequences with categorical (co)limits)**.** Denote the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ which we consider as a poset category. Let X be a set and $f : X \to \overline{\mathbb{R}}$ a function. We will see f as a diagram of shape X in $\overline{\mathbb{R}}$ by considering X as a discrete category, *i.e* a category whose only morphisms are the identity morphisms. The limits and colimits of such diagrams then exist and are respectively $\inf_{x\in X}(f(x))$ and $\sup_{x\in X}(f(x))$.

We can use this setting to describe limits of sequences. Let $x = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. It can also be written as a map $\mathbb{N} \xrightarrow{x} \overline{\mathbb{R}}$. Consider the diagram

$$F : \mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N} \xrightarrow{x} \mathbb{R}.$$

Then we have

$$\begin{aligned} \operatorname*{colim}_n \lim_m x_{n+m} &= \sup_{n\geq 0} \inf_{m\geq 0} x_{n+m} = \liminf_{n\to+\infty} x_n \\ \lim_n \operatorname*{colim}_m x_{n+m} &= \inf_{n\geq 0} \sup_{m\geq 0} x_{n+m} = \limsup_{n\to+\infty} x_n. \end{aligned}$$

By Lemma 11 we have

$$\liminf_n x_n \leq \limsup_n x_n$$

and we know that the equality holds if and only if x admits a limit.

We thus have many examples where limits and colimits do not commute. To conclude this section we give, without proof, a setting in which limits commute with certain colimits just to show that such things exist.

Definition 13. A small category I is filtered if there is a cone under every finite diagram in I.

Theorem 14. *Filtered colimits commute with finite limits in Set.*