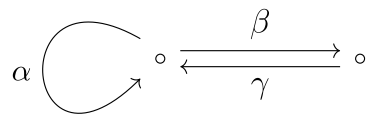
Exercise session: week 8

December 3, 2024

Let R denote a commutative ring; $\Bbbk$ a field and A a k-algebra. Exercises marked with $\star$ can be handed in for grading.

Exercise 1. Let $Q = (Q_0, Q_1)$ be the quiver



1. Which of the following ideals of $\Bbbk Q$ is admissible?

$$I_1 = \langle \alpha^2 - \beta\gamma, \gamma\beta - \gamma\alpha\beta, \alpha^4 \rangle, \qquad I_2 = \langle \alpha^2 - \beta\gamma, \gamma\beta, \alpha^4 \rangle,$$
$$I_3 = \langle (\gamma\beta)^4, \alpha^3 \rangle, \qquad I_4 = \langle \beta\gamma\alpha - \beta\gamma, (\beta\gamma)^2, \alpha^2 \rangle$$

2. Out of the admissible ideals, which pairs yield isomorphic quotients of the algebra $\Bbbk Q$? The answer might depend on the characteristic of the field $\Bbbk$.

$\star$ **Exercise 2.** Describe, up to isomorphism, all basic three dimensional algebras.

Exercise 3. Let Q be a finite quiver. Show that:

1. $\Bbbk Q$ is semisimple is and only if $|Q_1| = 0$,

2. $\Bbbk Q$ is simple if and only if $|Q_0| = 1$ and $|Q_1| = 0$.

Of moreover Q is connected, show that:

3. $\Bbbk Q$ is local only if $|Q_0| = 1$ and $|Q_1| = 0$.

4. $\Bbbk Q$ is commutative if and only if $|Q_0| = 1$ and $|Q_1| \leq 1$.

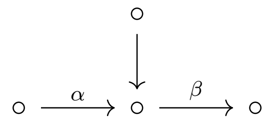
Exercise 4. Let A be an algebra such that $\operatorname{rad}^2 A = 0$. Show that if $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents then $e_i A e_j \neq 0$ if and only if there exists an arrow $i \to j$ in Q_A.

Exercise 5. Write a bound quiver presentation of each of the following algebras:

$$\begin{pmatrix} \Bbbk & 0 & 0 & 0 & 0 \\ \Bbbk & \Bbbk & 0 & 0 & 0 \\ \Bbbk & 0 & \Bbbk & 0 & 0 \\ \Bbbk & 0 & \Bbbk & \Bbbk & 0 \\ \Bbbk & \Bbbk & \Bbbk & \Bbbk & \Bbbk \end{pmatrix}, \quad \begin{pmatrix} \Bbbk & 0 & 0 & 0 & 0 \\ \Bbbk & \Bbbk & 0 & 0 & 0 \\ \Bbbk & 0 & \Bbbk & 0 & 0 \\ \Bbbk & 0 & 0 & \Bbbk & 0 \\ \Bbbk & \Bbbk & \Bbbk & \Bbbk & \Bbbk \end{pmatrix}, \quad \begin{pmatrix} \Bbbk & 0 & 0 & 0 & 0 \\ \Bbbk & \Bbbk & 0 & 0 & 0 \\ \Bbbk & \Bbbk & \Bbbk & 0 & 0 \\ \Bbbk & 0 & 0 & \Bbbk & 0 \\ \Bbbk & \Bbbk & \Bbbk & \Bbbk & \Bbbk \end{pmatrix}.$$

⋆ **Exercise 6.** Consider the quiver:

$Q :$



and the ideal $< \alpha\beta >$. Describe the simple modules, the indecomposable projectives and the indecomposable injective modules over the algebra $\Bbbk Q/I$ using quiver representations.