Exercise session: week 1

October 11, 2024

Let R denote a commutative ring; k a field and A a k-algebra. Exercises marked with $\star$ can be handed in for grading.

Exercise 1. Let R be a commutative ring. Show that the polynomial ring in two commutatingvariables $R[X, Y] := (R[X])[Y]$ is a commutative ring. Similarly, argue that the polynomials in n commuting variables $x_1, \ldots, x_n$ with coefficients in R, written $R[x_1, \ldots, x_n]$ also form a commutative ring.

$\star$ **Exercise 2.** Let B be a subset of A. Show that B is a k-subalgebra of A if and only if B is itself a k-algebra with the operation induced from A.

$\star$ **Exercise 3.** Let I be a two-sided ideal of a k-algebra A. Show that quotient k-vector space A/I has a unique k-algebra structure such that the canonical surjective map $\pi : A \to A/I, a \mapsto \overline{a} := a + I$, becomes a k-algebra homomorphism.

Exercise 4. Prove the following assertions.

(i) A (left, right or two sided) ideal I of a k-algebra A contains 1 if and only if $I = A$.

(ii) Let B be a subalgebra of A. B is a left (or right) ideal if and only if $B = A$.

$\star$ **Exercise 5.** Show that the only two sided ideals of $M_n(k)$ are the trivial ideals.

Exercise 6. Let n be an integer. Check that the ring $k[x_1, \ldots, x_n]$ is an infinite dimensional k-algebra.

Exercise 7. (i) Consider the incidence algebra kI of the poset $I = \{1 \succ 3 \prec 2\}$. Show that

$$kI \cong \left\{ \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ k & k & k \end{pmatrix} \right\}$$

as a k-algebra.

(ii) Consider the poset J defined by $\{3 \succ 4 \prec 2 \prec 1 \succ 3\}$. Show that

$$kJ \cong \left\{ \begin{pmatrix} k & 0 & 0 & 0 \\ k & k & 0 & 0 \\ k & 0 & k & 0 \\ k & k & k & k \end{pmatrix} \right\}$$

as a k-algebra.

Exercise 8. (i) Let A_1 and A_2 be k-algebras. Show that the direct product $A_1 \times A_2$ of A_1 with A_2, equipped with term wise multiplication, is a k-algebra with identity element $1 = (1, 1) = e_1 + e_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Do the same for the product of n k-algebras.

(ii) Let A be a k-algebra. Prove that the opposite algebra A^{op} is a a k-algebra. Then prove that $(A^{op})^{op} = A$.

⋆ **Exercise 9.** Show that $\operatorname{rad}(A/\operatorname{rad} A) = 0$.

⋆ **Exercise 10.** Let M, N and N be right A-modules. Check that the composition $\circ$,

$$\begin{array}{ccc} \operatorname{Hom}_A(M, N) \times \operatorname{Hom}_A(L, M) & \to & \operatorname{Hom}_A(L, N) \\ (h, g) & \mapsto & h \circ g \end{array}$$

is k-bilinear.