Exercise session: week 2

October 22, 2024

Let R denote a commutative ring; k a field and A a k-algebra. Exercises marked with $\star$ can be handed in for grading.

Exercise 1. Let M, N and N be right A-modules. Check that the composition $\circ$,

$$\begin{array}{ccc} \mathrm{Hom}_A(M,N) \times \mathrm{Hom}_A(L,M) & \to & \mathrm{Hom}_A(L,N) \\ (h,g) & \mapsto & h \circ g \end{array}$$

is k-bilinear.

$\star$ **Exercise 2** ($\bigotimes$ - Hom adjunction)**.** Before attempting this exercise, we invite you to consult the definition of the tensor product functors in the notes (Example 1.39). Let M be an $A-B$ bimodule, let X be right A-module and let Y be a right B-module. Consider the following maps.

$$\begin{array}{cccccc} & \mathrm{Hom}_{\mathrm{Mod}\,B}(X \otimes_A M, Y) & \longleftrightarrow & \mathrm{Hom}_{\mathrm{Mod}\,A}(X, \mathrm{Hom}_{\mathrm{Mod}\,B}(M,Y)) & \\ \Phi_{X,Y} : & f & \longmapsto & (x \mapsto (m \mapsto f(x \otimes m))) & \\ & (x \otimes m \mapsto \phi(x)(m)) & \longleftarrow & \phi & : \Psi_{X,Y} \end{array}$$

(i) Check that $\Phi_{X,Y}$ and $\Psi_{X,Y}$ are well defined

(ii) Show that $\Phi_{X,Y}$ and $\Psi_{X,Y}$ are inverse of one another

(iii) Let $\lambda : X \to X'$ be a morphism of right A-modules and $\mu : Y \to Y'$ be a morphism of left B modules. Show that the following diagrams commute.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Mod}\,B}(X \otimes_A M, Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Hom}_{\mathrm{Mod}\,A}(X, \mathrm{Hom}_{\mathrm{Mod}\,B}(M,Y)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Mod}\,B}(X \otimes_A M, Y') & \xrightarrow{\Phi_{X,Y'}} & \mathrm{Hom}_{\mathrm{Mod}\,A}(X, \mathrm{Hom}_{\mathrm{Mod}\,B}(M,Y')) \end{array} \tag{1}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Mod}\,B}(X' \otimes_A M, Y) & \xrightarrow{\Phi_{X',Y}} & \mathrm{Hom}_{\mathrm{Mod}\,A}(X', \mathrm{Hom}_{\mathrm{Mod}\,B}(M,Y)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Mod}\,B}(X \otimes_A M, Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Hom}_{\mathrm{Mod}\,A}(X, \mathrm{Hom}_{\mathrm{Mod}\,B}(M,Y)) \end{array} \tag{2}$$

where the vertical maps are obtained by applying the correct functors to the maps μ and λ respectively.

This shows that $-\bigotimes_A M$ and $\mathrm{Hom}_B(M,-)$ form a pair of adjoint functors. You can find the general definition here for example.

Exercise 3. Let $f : A \to B$ be a morphism of algebras.

(i) Let M be a right B-module. Show that, for all $m \in M$ and $a \in A$ setting $m \star a = m \cdot f(a)$ defines a right A-module structure on M.

(ii) Using this operation, construct a functor $Res : \mathrm{Mod}\, B \to \mathrm{Mod}\, A$. We call this the restriction functor.

(iii) Show that this functor is right adjoint to the functor

$$-\bigotimes_A {}_A B : \mathrm{Mod}\, A \to \mathrm{Mod}\, B.$$

This is the induction functor from A to B.

(iv) From now on let B be $T_2(k)$, let A be the subalgebra of diagonal matrices of B and let f be the inclusion $A \to B$. Show that k^2 is an indecomposable module over B but that $Res(k^2)$ is not indecomposable over A.

(v) Decompose B into indecomposable components as an A-module.

Exercise 4. Let $f : A \to B$ be a homomorphism of k-algebras.

(i) Show that if f is surjective then $f(\mathrm{rad}\, A) \subseteq \mathrm{rad}\, B$.

(ii) Find a counterexample when f is not surjective. *Hint: localise a polynomial algebra at a prime ideal and map it to its ring of fractions.*

Exercise 5. Let A be the polynomial k-algebra $k[t_1, t_2]$.

(i) Recall that a ring said to be local if it has a unique maximal ideal. Prove that A is not local.

(ii) Prove that the elements 0 and 1 are the only idempotents of A.

(iii) Prove that the radical of A is zero.

⋆ **Exercise 6.** Let A be the algebra $T_2(k)$ of lower triangular matrices in $M_2(k)$. Consider the two vector spaces $P_1 = \{[a_1, 0] | a_1 \in k\}$ and $P_2 = \{[a_1, a_2] | a_i \in k\}$.

(i) Show that they have a right $T_2(k)$-module structure.

(ii) Show that the inclusion $\phi : P_1 \hookrightarrow P_2$ is a homomorphism of $T_2(k)$ modules.

(iii) Cokernel of ϕ is a right $T_2(k)$-module. We denote it S_2[1]. Describe it.

(iv) Show that P_1, P_2 and S_2 are indecomposable modules.

[1] The reason behind that choice will become apparent in the following weeks.

Exercise 7. The goal of this exercise is to show that right $T_2(k)$-modules from the previous exercise P_1, P_2 and S_2, are the only indecomposable finite dimensional right modules over $T_2(k)$. Let M be a right $T_2(k)$ module.

(i) Using the actions of the basis elements of $T_2(k)$, E_{11}, E_{22} and E_{21} decompose M into two subvector spaces linked by a linear map

$$V_2 \xrightarrow{f} V_1$$

(ii) As vector spaces, we can decompose V_1 as $\operatorname{coker}(f) \oplus \operatorname{Im}(f)$ and V_2 as $\ker(f) \oplus V_2'$. Show that $\operatorname{coker}(f), \ker(f)$ and $\operatorname{Im}(f) \oplus V_2'$ are sub representations of M.

(iii) Show that they are in direct sum.

(iv) Show that $\operatorname{coker}(f) \cong (S_2)^{\dim \operatorname{coker}(f)}$ as modules.

(v) Show that $\ker(f) \cong (P_1)^{\dim \ker(f)}$ as modules.

(vi) Show that $\dim \operatorname{Im}(f) = \dim V_2'$.

(vii) Show that $\operatorname{Im}(f) \oplus V_2' \cong P_2^{\dim \operatorname{Im}(f)}$ as modules.

(viii) Conclude that M is indecomposable if and only if $M \cong P_1$, $M \cong P_2$ or $M \cong S_2$.

⋆ **Exercise 8.** Les $\mathfrak{S}_3$ be the symmetric group on three elements and let A be its group algebra $k\mathfrak{S}_3$.

(i) Show that setting $g \cdot x = x$ for all $g \in \mathfrak{S}_3$ and $x \in k$ defines a left A module structure on k. We call this the trivial module of A.

(ii) Denote $sn : \mathfrak{S}_3 \to \{-1, 1\}$ the sign map on $\mathfrak{S}_3$. Show that setting $g \cdot x = sn(g) \cdot x$ defines another left A-module structure on k.

(iii) Recall that for each permutation $g \in \mathfrak{S}_3$ we can associate a permutation matrix defined $\sum_{i=1}^{3} E_{i\sigma(i)}$. Show that sending g to its permutation matrix P_g defines a left A-module structure on the subvector space $\{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\}$ of k^3.

(iv) Suppose the field k is $\mathbb{C}$. Show that these three modules are indecomposable.

(v) What can happen when k is not $\mathbb{C}$?

Next week we will show that these are the only indecomposable left modules over $\mathbb{C}$ and that they are even irreducible.