Exercise session: week 3

October 30, 2024

Let R denote a commutative ring; k a field and A a k-algebra. Exercises marked with $\star$ can be handed in for grading.

$\star$ **Exercise 1.** Let M and N be right A-modules. Let $f : M \to N$ be a right A-module homomorphisms.

1. Show that $\operatorname{rad}(M \oplus N) \cong \operatorname{rad} M \oplus \operatorname{rad} N$.
2. Show that $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$.
3. Give an example where the above inequality is strict.

$\star$ **Exercise 2.** Let $A = M_n(k)$ with $n > 1$

1. Find orthogonal idempotents in A.
2. Find primitive idempotents in A.
3. Think about whether or not you have found all of them.

$\star$ **Exercise 3.** Consider the set of matrices

$$\begin{pmatrix} \mathbb{k} & 0 & 0 \\ \mathbb{k} & \mathbb{k} & \mathbb{k} \\ \mathbb{k} & \mathbb{k} & \mathbb{k} \end{pmatrix}$$

1. Show that it is a subalgebra of $M_n(\mathbb{k})$. Write it A.
2. Show that E_{11}, E_{22} and E_{33} form a complete set of idempotents.
3. Show that the projectives $E_{22} \cdot A$ and $E_{33} \cdot A$ are isomorphic. *Indication:* consider left multiplication by well chosen matrices $E_{i,j}$ that belong to A.
4. Show that $\operatorname{Hom}_A(E_{11} \cdot A, E_{22} \cdot A) = 0$. Deduce that $E_{11} \cdot A \ncong E_{22} \cdot A$.
5. Set. $P = E_{11} \cdot A \oplus E_{22} \cdot A$. Compute $\operatorname{End}_A(P)$. *Indication:* Think in terms of matrices, use the previous questions and compute $\operatorname{Hom}_A(E_{22} \cdot A, E_{11} \cdot A)$.

Exercise 4. Let G be a finite group. Let k be a field. Suppose that $|G|$ is invertible in k. Prove that the group algebra kG is semisimple (Maschke's theorem). Here is suggestion for the proof [1]. Let M be a right kG-module We want to show that every kG-submodule N of M has a complement. Write e the element of $\mathrm{End}_k(M)$ which is a projection onto the subvector space N. Define $e' = \frac{1}{|G|}\sum_{g\in G} g \cdot e \cdot g^{-1}$

1. Justify that e' is well defined and that it is an idempotent of $\mathrm{End}_k(M)$.

2. Show that e' is an element of $\mathrm{End}_{kG}(M)$, $i.e$ that it commutes with the action of G. It follows that $(1-e')M$ is a submodule of M.

3. Show that $x \in N$ is an element of N if and only if that $e' \cdot x = x$.

4. Conclude the proof.

Exercise 5. In this exercise we give a guide to the proof of the Artin-Wedderburn Theorem. Let A be a finite dimensional algebra over an algebraically closed field k. The theorem states that the following assertions are equivalent.

(a) The right A-module A_A is semisimple.

(b) Every right A-module is semisimple.

(c) The left A-module ${}_AA$ is semisimple.

(d) Every left A-module is semisimple.

(e) $\mathrm{rad}\, A = 0$.

(f) There exist positive integers $m_1, \ldots, m_s$ and a k-algebra isomorphism

$$A \cong M_{m1}(k) \times \cdots \times M_{m_s}(k)$$

1. Show that

$$\begin{aligned} \mathrm{End}_{\mathrm{Mod}\, A}(A) &\to A^{op} \\ \phi &\mapsto \phi(1_A) \end{aligned}$$

is an isomorphism of k-algebras.

2. Suppose A is finite dimensional semisimple algebra. Using Lemma 1.41, write $\mathrm{End}_{\mathrm{Mod}\, A}(A)$ as a direct sum of homomorphism spaces between simple modules.

3. If S and S' are simple module, what are the possible values of $\mathrm{Hom}_{\mathrm{Mod}\, A}(S, S')$?

4. Deduce the implication $(a) \Rightarrow (d)$ of the theorem.

5. Suppose A satisfies condition (d). Show that $\mathrm{rad}\, A = 0$.

[1] Taken from these notes

6. Suppose that $\operatorname{rad} A = 0$. Let S be a simple right submodule of S. Show that there exists a maximal right ideal M of A such that $S \oplus M = A$.

7. Deduce that, when $\operatorname{rad} A = 0$, the socle of A is a direct summand of A.

8. Deduce that $(c) \Rightarrow (a)$.

9. Using Lemma 1.43 show that $(a) \Rightarrow (b)$.

Exercise 6. Prove or read the proof of the Jordan–Hölder theorem.