Extra homework: Posets, incidence algebras and noteworthy representations

November 28, 2024

Let $\Bbbk$ be a field and A a k-algebra.

Summary: Exercise 1. is about a presentation of the incidence algebra of a finite posets by a bound quiver. Exercise 3 is about the global dimension of the incidence algebra of a finite poset being bounded by the size of the poset. You can use Exercise 2 to prove that. Exercise 4, is about interval representations of incidence algebras.

Exercise 1. Let P be a finite set equipped with a partial order relation $\leq$. Let x, y be elements of P. We say that $x \leq y$ is a *covering relation* if the interval $[x, y]$ of the poset contains exactly two elements. Define the *Hasse diagram* H of the poset $(P, \leq)$ to be the directed graph with vertices the elements of P and arrows from x to y if and only if the relation $x \leq y$ holds and is a covering relation.

1. Consider the set $\{1, 2, 3, 4, 6, 12\}$ orderd by the relation $x \leq y$ if and only if x divides y. Draw its Hasse diagram. Compute its incidence algebra.

2. Show that the quiver algebra $\Bbbk H$ surjects onto the incidence algebra of the poset $(P, \leq)$. Give an explicit morphism of algebras.

3. Give a set of generators for the kernel of this map using paths in the Hasse diagram.

4. Show that this ideal is admissible.

Exercise 2. Suppose A is finite dimensional. Let N be a superfluous submodule of a projective finitely generated right A-module P. Show that $\operatorname{top} N$ cannot contain simples that appear in $\operatorname{top} P$.

Exercise 3. Let P be a finite set and $\leq$ be a partial order relation on P. Let A be the incidence algebra of P. For each element $x \in P$ we denote by e_x its associated idempotent and $P_x = e_x \cdot A$ its associate indecomposable projective module.

1. Show that $\operatorname{Hom}_A(P_x, P_y) \neq 0$ if and only if $y \leq x$.

2. Suppose that $y \leq x$. Compute $\dim \operatorname{Hom}_A(P_x, P_y)$.

3. Using the previous exercise, show the projective indecomposable summands of a minimal projective resolution must decrease (in the sens of the order relation of the poset) when the degree[1] increases.

4. Conclude that the global dimension of A is bounded by the size of P.

Exercise 4. Let P be a poset. Let a, c, b, d be elements of P such that $a \leq b$ and $c \leq d$. Let $H = (Q_0, Q_1)$ be the Hasse diagram of P.

1. Show that the following data defines a bound quiver representation for H:

$$M_i = \begin{cases} \Bbbk & \text{if } i \in [a,b] \\ 0 & \text{otherwise,} \end{cases} ; \qquad \phi_\alpha = \begin{cases} id_\Bbbk & \text{if } s(\alpha), t(\alpha) \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

for $i \in Q_0$ and $\alpha \in Q_1$. This is the representation associated to the interval $[a,b] = \{c \in P | a \leq c \leq b\}$ of P. Denote it $M_{[a,b]}$.

2. Describe the interval module $M_{[1,6]}$ for the poset from Exercise 1.1.

3. Show that there exists a non zero morphism of representations from $M_{[a,b]}$ to $M_{[c,d]}$ if and only if the following inequalities hold

$$c \leq a \leq d \leq b.$$

Note: Projective indecomposable, injective indecomposable and simple modules of incidence algebra of a poset are all interval modules.

[1] the degree of a summand is its position in the projective resolution