Exercise session: week 10

December 12, 2024

Let R denote a commutative ring; $\Bbbk$ a field and A a k-algebra. Exercises marked with $\star$ can be handed in for grading.

$\star$ **Exercise 1.** Describe, up to isomorphism, all basic three dimensional algebras.

$\star$ **Exercise 2.** Consider the quiver:

$$Q:\quad \begin{array}{ccccc} & & \circ & & \\ & & \downarrow & & \\ \circ & \xrightarrow{\alpha} & \circ & \xrightarrow{\beta} & \circ \end{array}$$

and the ideal $< \alpha\beta >$. Describe the simple modules, the indecomposable projectives and the indecomposable injective modules over the algebra $\Bbbk Q/I$ using quiver representations.

Exercise 3. Let $M = (M_\alpha, \phi_\alpha)$ be a $\Bbbk$-linear representation of the bound quiver (Q, I). The **support** $\operatorname{supp} M$ of M is the full subquiver of Q such that $(\operatorname{supp} M)_0 = \{b \in Q_0 | M_b \neq 0\}$.

1. Show that is M is indecomposable then $\operatorname{supp} M$ is connected.

2. Show that the converse is not true.

Exercise 4. Let Q be a finite quiver with at least one cycle. Show that the path algebra $A = \Bbbk Q$ has infinitly many pairwise non isomorphic simple modules of finite dimension.

Exercise 5. Revisit exercises 6 and 7 from week 2. Knowing that $T_2(\Bbbk)$ is isomorphic to the path algebra $\Bbbk(\circ \to \circ)$, how can these two exercises be reformulated using what we now know about quivers and quiver representations?

Exercise 6. Consider the quiver

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 2 & \longrightarrow & 5 & \longleftarrow & 4 \\ & & \uparrow & & \\ & & 3 & & \end{array}$$

Let λ, μ be in $\mathbb{k}^\times$. Denote M_λ the quiver representation

$$\begin{array}{ccccc} & & \mathbb{k} & & \\ & & \big\downarrow{\scriptstyle (1,1)} & & \\ \mathbb{k} & \xrightarrow{(0,1)} & \mathbb{k}^2 & \xleftarrow[(1,0)]{} & \mathbb{k} \\ & & {\scriptstyle (1,\lambda)}\big\uparrow & & \\ & & \mathbb{k} & & \end{array}$$

1. What is the dimension vector v of this representation?

2. Show that $M_\mu \cong M_\lambda$ if and only if $\lambda = \mu$.

3. Show that there is a bijection between triples of proper subspaces of a fixed two dimensional vector space and representations of the quiver Q with dimension vector v

4. Deduce that there are infinitely many triples of proper subspaces of a fixed two dimensional vector space.

Exercise 7. Consider the following quiver Q

$$\begin{array}{ccccc} & & 1 & & \\ & & \big\downarrow & & \\ 2 & \longrightarrow & 4 & \longleftarrow & 3 \end{array}$$

Let λ, μ be in $\mathbb{k}$. Denote M_λ the quiver representation

$$\begin{array}{ccccc} & & \mathbb{k} & & \\ & & \big\downarrow{\scriptstyle (1,0)} & & \\ \mathbb{k} & \xrightarrow{(0,1)} & \mathbb{k}^2 & \xleftarrow[(1,\lambda)]{} & \mathbb{k} \end{array}$$

1. What is the dimension vector v of this representation?

2. Show that $M_\mu \cong M_\lambda$ for all $\lambda, \mu \in \mathbb{k}^\times$.

3. Compare this with the result of the previous exercise.