

Exercise session: week 11

December 12, 2024

Let R denote a commutative ring; \mathbb{k} a field and A a k -algebra. Exercises marked with \star can be handed in for grading.

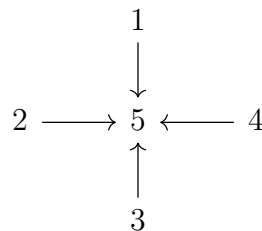
Exercise 1. Let $M = (M_\alpha, \phi_\alpha)$ be a \mathbb{k} -linear representation of the bound quiver (Q, I) . The **support** $\text{supp } M$ of M is the full subquiver of Q such that $(\text{supp } M)_0 = \{b \in Q_0 \mid M_b \neq 0\}$.

1. Show that if M is indecomposable then $\text{supp } M$ is connected.
2. Show that the converse is not true.

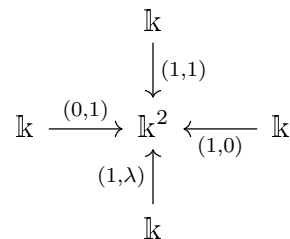
Exercise 2. Let Q be a finite quiver with at least one cycle. Show that the path algebra $A = \mathbb{k}Q$ has infinitely many pairwise non isomorphic simple modules of finite dimension.

Exercise 3. Revisit exercises 6 and 7 from week 2. Knowing that $T_2(\mathbb{k})$ is isomorphic to the path algebra $\mathbb{k}(\circ \rightarrow \circ)$, how can these two exercises be reformulated using what we now know about quivers and quiver representations?

Exercise 4. Consider the quiver



Let λ, μ be in \mathbb{k}^\times . Denote M_λ the quiver representation



1. What is the dimension vector v of this representation?

2. Show that $M_\mu \cong M_\lambda$ if and only if $\lambda = \mu$.
3. Show that there is a bijection between triples of proper subspaces of a fixed two dimensional vector space and representations of the quiver Q with dimension vector v
4. Deduce that there are infinitely many triples of proper subspaces of a fixed two dimensional vector space.

Exercise 5. Let $r \geq 4$ be an integer. Let λ be in \mathbb{k} . Consider the quiver $Q_r = (Q_0, Q_1)$ defined by

$$Q_0 = \{1, 2, \dots, r, r+1\}, \quad Q_1 = \{a_i : i \rightarrow r+1 \mid 1 \leq i \leq r\}. \quad (1)$$

Extend the definition of the modules M_λ from the previous exercise so as to construct infinite families of non isomorphic quiver representations over each Q_r .

Exercise 6. Consider the following quiver Q

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 2 & \longrightarrow & 4 & \longleftarrow & 3 \end{array}$$

Let λ, μ be in \mathbb{k} . Denote M_λ the quiver representation

$$\begin{array}{ccccc} & & \mathbb{k} & & \\ & & \downarrow (1,0) & & \\ \mathbb{k} & \xrightarrow{(0,1)} & \mathbb{k}^2 & \xleftarrow{(1,\lambda)} & \mathbb{k} \end{array}$$

1. What is the dimension vector v of this representation?
2. Show that $M_\mu \cong M_\lambda$ for all $\lambda, \mu \in \mathbb{k}^\times$.
3. Compare this with the result of the previous exercise.