Exercise session: week 3
Erratum

October 30, 2024

Let R denote a commutative ring; k a field and A a k-algebra. This file contains solutions for two questions from the exercise sheet on which some false remarks were made during the exercise class.

Exercise 1. Let M and N be right A-modules.

1. Show that $\operatorname{rad}(M \oplus N) \cong \operatorname{rad} M \oplus \operatorname{rad} N$.

Solution.

Set up: Recall that a direct sum of modules $M \oplus N$ has canonical injections attached to it

$$\iota_M : M \to M \oplus N \text{ and } \iota_N : N \to M \oplus M.$$

Let $f : M \oplus N \to S$ be right A-module homomorphism. Consider the homomorphisms $f_M = f \circ \iota_M$ and $f_N = f \circ \iota_N$. Let $x \oplus y$ be an element of $M \oplus N$. Because f is a linear, we have

$$f(x \oplus y) = f_M(x) + f_N(y) \text{ in } S.$$

First inclusion: First we show that $\operatorname{rad}(M \oplus N) \subseteq \operatorname{rad} M \oplus \operatorname{rad} N$. Suppose $x \oplus y$ is an element of $\operatorname{rad}(M \oplus N)$. By Proposition 1.49.a), for all simple module S and map f as above, we have $f(x \oplus y) = 0$. Because we can take any map f, we can in particular choose a map $f_M : M \to S$ and define f be setting

$$f(x_1 \oplus x_2) = f_M(x_1).$$

In that case, $f(x \oplus y) = 0$ implies that $f_M(x) = 0$. Because this is true for any choice of homomorphism $f_M : M \to S$, it follows that $x \in \operatorname{rad} M$. A similar argument shows that $y \in \operatorname{rad} N$. This proves the inclusion $\operatorname{rad}(M \oplus N) \subseteq \operatorname{rad} M \oplus \operatorname{rad} N$.

Second inclusion: To show that $\operatorname{rad} M \oplus \operatorname{rad} N \subseteq \operatorname{rad}(M \oplus N)$, let S be a simple module, let $f : M \oplus N \to S$ a homomorphism and take $x \in \operatorname{rad} M$ as well as $y \in \operatorname{rad} N$. We apply Proposition 1.49.a) to $\operatorname{rad} M$ and the maps $f_M = f \circ \iota_M : M \to S$ and to $radN$ and the map $f_N = f \circ \iota_N : N \to S$ respectively. We get $f_M(x) = 0 = f_N(y)$ in S. It follows that

$$f(x \oplus y) = f_M(x) + f_N(y) = 0.$$

By applying Proposition 1.49.a) to $\operatorname{rad}(A \oplus N)$ we get $x \oplus y \in \operatorname{rad}(M \oplus N)$. This concludes the second inclusion and the proof of the property.

Remark: In this proof we used the universal property of the direct product. Feel free to look up what it means or to ask questions about it. About the mistake made during the examples class: what I was trying to prove about the shape of the maximal ideals is false. If we take $A = k$ acting on the module $k \oplus k$, then the diagonal submodule defined by sending x to $x \oplus x$ is maximal.

Exercise 2. Let $A = M_n(k)$ with $n > 1$

2. Having identified the idempotents E_{ii}, we wanted to show that they are primitive.

Set up: To do so assume we can write $E_{ii} = e + e'$ were e and e' are idempotents of A. We write $(e_{kl})_{k,l\leq n}$ the entries of e and $(e'_{kl})_{k,l\leq n}$ the entries of e'.

First reduction: Because E_{ii} has only one non zero entry, we can already deduce that $e_{kl} = -e'_{kl}$ for all k, l such that $k \neq i$ or $l \neq i$. Moreover, $e_{ii} + e'_{ii} = 1$.

Second reduction: We now write $e = E_i i - e'$. Because e, e' and E_{ii} are idempotents, we have

$$e = E_{ii} + e' - E_{ii} \cdot e' - e' \cdot E_{ii}.$$

Observe that the matrix $E_{ii} \cdot e'$ has no nonzero entries outside of line i. Similarly, the matrix $e' \cdot E_{ii}$ has no nonzero entries outside of column i. Hence, when $k \neq i$ or $l \neq i$ we get $e'_{kl} = -e'_{kl}$ (notice the e' on both sides and not e). So for such k and l we have $e_{kl} = e'_{kl} = 0$. We have showed that the idempotents e and e' must have non zero entries only in line i and column i. You can think of two lines crossing at a right angle on the diagonal.

Third reduction, case 1: Suppose there exists $l \leq n$ such that $e_{il} \neq 0$. Then the equality

$$e_{il} = \sum_{k=1}^{n} e_{ik} \cdot e_{kl} = e_{ii} \cdot e_{il}$$

implies that $e_{ii} = 1$. Moreover,

$$e_{kl} = e_{ki} \cdot e_{il}$$

implies that $e_{ki} = 0$ when $k \neq l$. This shows that e and e' have all their non zero entries on line i. A computation shows that $e'_{ii} = 0$ implies that $e' = 0$. Hence $e = E_{ii}$ but this contradicts $e_{il} \neq 0$.

Third reduction, case 2: Suppose there exists $k \leq n$ such that $e_{ki} \neq 0$. A similar argument leads a contradiction again.

Conclusion The matrices e and e' have at most one nonzero entry and it is e_{ii} or e'_{ii}. The field k has only idempotents 0 and 1, so one of these two idempotents is equal to E_{ii} and the other is zero. This concludes the proof.

3. The diagonal idempotents E_I which we defined for $I \subseteq \{1, \ldots, n\}$ **are far from being the only idempotents** in $A = M_n(k)$. Below are concrete examples in $M_2(k)$, but feel free to find some in higher dimension. I don't have a complete list but maybe it exists somewhere.

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$