# Exercise session: week 4

November 5, 2024

Let $R$ denote a commutative ring; $k$ a field and $A$ a $k$-algebra. Exercises marked with $\star$ can be handed in for grading.

**Exercise 1.** Show that a subset $I$ of $A$ is a (left, right, - ) ideal of $A$ if and only if it is a (left, right, -) $A$-submodule.

$\star$ **Exercise 2.** Consider the set of matrices

$$\begin{pmatrix} \Bbbk & 0 & 0 \\ \Bbbk & \Bbbk & \Bbbk \\ \Bbbk & \Bbbk & \Bbbk \end{pmatrix}$$

1. Show that it is a subalgebra of $M_n(\Bbbk)$. Write it $A$.

2. Show that $E_{11}, E_{22}$ and $E_{33}$ form a complete set of primitive idempotents.

3. Show that the modules $E_{22} \cdot A$ and $E_{33} \cdot A$ are isomorphic. *Indication:* consider left multiplication by well chosen matrices $E_{i,j}$ that belong to $A$.

4. Show that $\mathrm{Hom}_A(E_{22} \cdot A, E_{11} \cdot A) = 0$. Deduce that $E_{11} \cdot A \ncong E_{22} \cdot A$.

5. Set $P = E_{22} \cdot A \oplus E_{11} \cdot A$. Compute $\mathrm{End}_A(P)$. *Indication:* Think in terms of matrices, use the previous questions and compute $\mathrm{Hom}_A(E_{11} \cdot A, E_{22} \cdot A)$.

$\star$ **Exercise 3.** Let $r : M \to N$ be a homomorphism of right $A$-modules. Show that $r$ admits a section if and only if $r$ is surjective and $M = L \oplus \ker r$ where $L$ is a submodule of $M$.

**Exercise 4.** Let $u : L \to M$ be a homomorphism of right $A$-modules. Show that $u$ admits a retraction if and only if $u$ is injective and $M = \mathrm{Im}(u) \oplus N$ where $N$ is a submodule of $M$.

**Exercise 5.** Suppose that the sequence $0 \to L \xrightarrow{u} M \xrightarrow{r} N \to 0$ of right $A$-modules is exact. Prove that the homomorphism $u$ admits retraction if and only if $r$ admits as section.

**Exercise 6.** Let $Z(A)$ denote the centre of $A$. Show that the following propositions are equivalent.

1. The algebra $A$ is connected.

2. The algebra $Z(A)$ is connected.

3. The elements 0 and 1 are the only central idempotents of $A$.

**Exercise 7.** Let $e$ be a primitive idempotent of $A$. Show that the algebra $eAe$ is local.