Exercise session: week 5

November 6, 2024

Let R denote a commutative ring; k a field and A a k-algebra. Exercises marked with $\star$ can be handed in for grading.

$\star$ **Exercise 1.** Let $r : M \to N$ be a homomorphism of right A-modules. Show that r admits a section if and only if r is surjective and $M = L \oplus \ker r$ where L is a submodule of M.

Exercise 2. Let $u : L \to M$ be a homomorphism of right A-modules. Show that u admits a retraction if and only if u is injective and $M = \mathrm{Im}(u) \oplus N$ where N is a submodule of M.

Exercise 3. Suppose that the sequence $0 \to L \xrightarrow{u} M \xrightarrow{r} N \to 0$ of right A-modules is exact. Prove that the homomorphism u admits retraction if and only if r admits as section.

Exercise 4. Let M be a right A-module and let $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ be a short exact sequence.

1. Show that the sequence $0 \to \mathrm{Hom}_A(M, K) \xrightarrow{f\circ} \mathrm{Hom}_A(M, L) \xrightarrow{g\circ} \mathrm{Hom}_A(M, N)$ is exact. We say that the functor $\mathrm{Hom}_A(M, ?)$ is left exact.

2. Show that the sequence $0 \to \mathrm{Hom}_A(N, M) \xrightarrow{\circ g} \mathrm{Hom}_A(L, M) \xrightarrow{\circ f} \mathrm{Hom}_A(K, M)$ is exact. We say that the contravariant functor $\mathrm{Hom}_A(?, M)$ is right exact.

A functor that is both left and right exact is exact.

$\star$ **Exercise 5.** Let P be a right A-module and M a right A-module. Show that the following propositions are equivalent

1. P is projective.
2. Every surjective morphism $M \to P$ admits a section.
3. There exists a free right A-module F and a right module P' such that $P \oplus P' \cong F$.
4. The functor $\mathrm{Hom}_A(P, ?)$ is exact.
5. There exists a set $\{a_i \in P | i \in I\}$ and a set $\{f_i \in \mathrm{Hom}_A(P, A) | i \in I\}$ such that for $x \in P$, $f_i(x)$ is non zero for only finitely many elements $i \in I$ and $x = \sum_{i \in I} f_i(x) \cdot a_i$.

Exercise 6. With the help of Definition 1.5.1 from the lecture adapt items 1, 2 and 4 of exercise 5 to obtain similar characterisations for injective modules and prove their equivalence. *Note:* Items 3 and 5 do not have their injective counterparts.

Exercise 7. Show that the algebra $T_n(\mathbb{k})$ is basic.

Exercise 8. Suppose A is finite dimensional. Let N be a superfluous submodule of a projective finitely generated right A-module P. Show that $\operatorname{top} N$ cannot contain simples that appear in $\operatorname{top} P$.

Exercise 9. Let P be a finite set and $\leq$ be a partial order relation on P. Let A be the incidence algebra of P. For each element $x \in P$ we denote by e_x its associated idempotent and $P_x = e_x \cdot A$ its associate indecomposable projective module.

1. Show that $\operatorname{Hom}_A(P_x, P_y) \neq 0$ if and only if $y \leq x$.

2. Suppose that $y \leq x$. Compute $\dim \operatorname{Hom}_A(P_x, P_y)$.

3. Using the previous exercise, show the projective indecomposable summands of a minimal projective resolution must decrease (in the sens of the order relation of the poset) when the degree[1] increases.

4. Conclude that the global dimension of A is bounded by the size of P.

[1]the degree of a summand is its position in the projective resolution