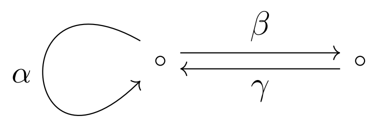
Exercise session: week 7

November 21, 2024

Let R denote a commutative ring; $\Bbbk$ a field and A a k-algebra. Exercises marked with $\star$ can be handed in for grading.

$\star$ **Exercise 1.** Let $Q = (Q_0, Q_1)$ be the quiver



1. Which of the following ideals of $\Bbbk Q$ is admissible?

$$I_1 = \langle \alpha^2 - \beta\gamma, \gamma\beta - \gamma\alpha\beta, \alpha^4 \rangle, \qquad I_2 = \langle \alpha^2 - \beta\gamma, \gamma\beta, \alpha^4 \rangle,$$
$$I_3 = \langle (\gamma\beta)^4, \alpha^3 \rangle, \qquad I_4 = \langle \beta\gamma\alpha - \beta\gamma, (\beta\gamma)^2, \alpha^2 \rangle$$

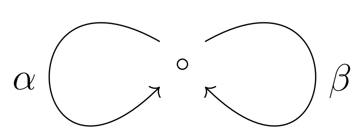
2. Out of the admissible ideals, which pairs yield isomorphic quotients of the algebra $\Bbbk Q$? The answer might depend on the characteristic of the field $\Bbbk$.

Exercise 2. Let Q be a quiver and I an admissible ideal of $\Bbbk Q$. Construct an admissible ideal I^{op} of $\Bbbk Q^{op}$ such that $\Bbbk Q^{op}/I^{op} \cong (\Bbbk Q/I)^{op}$.

$\star$ **Exercise 3.** Let $A = T_3(\Bbbk)$. Let C be the subalgebra of A consisting of matrices

$$\lambda = \begin{pmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

such that $\lambda_{11} = \lambda_{22} = \lambda_{33}$. Show that the algebra C is noncommutative, local and that there are $\Bbbk$-algebra isomorphisms $C \cong \Bbbk\langle t_1, t_2\rangle/(t_1^2, t_2^2, t_2t_1) = \Bbbk Q/I$ where Q is the quiver below and $I = \langle \alpha^2, \beta^2, \beta\alpha \rangle$.



Exercise 4. Let P be a finite set equipped with a partial order relation $\leq$. Let x, y be elements of P. We say that $x \leq y$ is a *covering relation* if the interval $[x, y]$ of the poset contains exactly two elements. Define the *Hasse diagram* of the poset $(P, \leq)$ to be the directed graph with vertices the elements of P and arrows from x to y if and only if the relation $x \leq y$ holds and is a covering relation.

1. Consider the set $\{1, 2, 3, 4, 6, 12\}$ orderd by the relation $x \leq y$ if and only if x divides y. Draw its Hasse diagram. Compute its incidence algebra.

2. Show that the quiver algebra $\Bbbk H$ surjects onto the incidence algebra of the poset $(P, \leq)$. Give an explicit morphism of algebras.

3. Give a set of generators for the kernel of this map using paths in the Hasse diagram.

4. Show that this ideal is admissible.